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PARAMETER CONTINUITY OF THE SOLUTIONS OF A MATHEMATICAL MODEL OF THERMOVISCOELASTICITY

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Abstract: In this paper the continuity of the solutions of a mathematical model of thermoviscoelasticity with respect to the model parameters is proved. This was an open problem conjectured in [27] and [28]. The nonlinear partial differential equations under consideration arise from the conservation laws of linear momentum and energy and describe structural phase transitions in solids with non-convex Landau-Ginzburg free energy potentials. The theories of analytic semigroups and real interpolation spaces for maximal accretive operators are used to show that the solutions of the model depend continuously on the admissible parameters, in particular, on those defining the free energy. More precisely, it is shown that if $\{q_n\}_{n=1}^{\infty}$ is a sequence of admissible parameters converging to q, then the corresponding solutions $z(t;q_n)$ converge to $z(t;q)$ in the norm of the graph of a fractional power of the operator associated to the linear part of the system.

1. INTRODUCTION

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The conservation laws governing the thermomechanical processes in a one-dimensional shape memory solid $\Omega = (0,1)$ with Landau-Ginzburg free energy potential Ψ give rise to the following initial-boundary value problem.

$$
(1.1)\begin{cases}
\rho u_{tt} - \beta\rho u_{xxt} + \gamma u_{xxxx} = f(x,t) + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \epsilon}\Psi(u_x, u_{xx}, \theta)\right], & x \in \Omega, 0 \le t \le T,\\
C_v\theta_t - k\theta_{xx} = g(x,t) + 2\alpha_2\theta u_x u_{xt} + \beta\rho u_{xt}^2, & x \in \Omega, 0 \le t \le T,\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), & x \in \Omega,\\
u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, & 0 \le t \le T,\\
\theta_x(0,t) = 0, \qquad k\theta_x(1,t) = k_1\left(\theta_\Gamma(t) - \theta(1,t)\right), & 0 \le t \le T.
\end{cases}
$$

The functions, variables and parameters involved in (1.1) have the following physical meaning: $u(x,t)$ = displacement; $\theta(x,t)$ = absolute temperature; ρ = mass density; k = thermal conductivity coefficient; C_v = specific heat; β = viscosity coefficient; $f(x,t)$ = distributed forces acting on the body (input); $g(x,t)$ = distributed heat sources (input); $u_0(x)$ = initial displacement; $u_1(x)$ = initial velocity; $\theta_0(x)$ = initial temperature; $\theta_\Gamma(t)$ = temperature of the surrounding medium (input); k_1 = positive constant, proportional to the rate of thermal exchange at the right boundary, and T is a prescribed final time. The function Ψ, which represents the free energy density of the system, is assumed to be a function of the linearized shear strain $\epsilon = u_x$, the spatial derivative of the strain $\epsilon_x = u_{xx}$ and the temperature θ, and is taken in the Landau-Ginzburg form

$$
\begin{aligned}
\Psi(\epsilon, \epsilon_x, \theta) &= \Psi_0(\theta) + \alpha_2(\theta - \theta_1)\epsilon^2 - \alpha_4\epsilon^4 + \alpha_6\epsilon^6 + \frac{\gamma}{2}\epsilon_x^2,\\
\Psi_0(\theta) &= -C_v\theta\log\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C,
\end{aligned}
\tag{1.2}
$$

where θ_1, θ_2 are two critical temperatures and α_2, α_4, α_6, γ are positive constants, all depending on the material being considered. Note that for values of θ close to θ_1 and ϵ_x fixed, the function $\Psi(\epsilon, \epsilon_x, \theta)$ is a nonconvex function of ϵ. This property is related to the hysteresis phenomenon which caracterizes this type of materials in the low and intermediate temperature ranges. The stress-strain relations are strongly temperature-dependent. The behavior goes from elastic, ideally-plastic at low temperatures, to pseudoelastic or superelastic at intermediate temperatures, to almost linearly elastic in the high temperature range. Shape memory and solid-solid phase transitions (martensitic transformations) are other peculiar characteristics of these materials whose dynamical behavior is described by system (1.1). For a detailed review of these and other properties and the derivations of the equations in (1.1) we refer the reader to [25] and the references therein.

The boundary conditions mean that the body is clamped at both ends, thermally insulated at the left end and, at the right end, the rate of thermal exchange is prescribed. The nonlinear coupled equations in (1.1) are sometimes referred to as the equations of thermo-visco-elasto-plasticity. In particular, the first equation in (1.1) can be regarded as a nonlinear beam equation in u, while the second is a nonlinear heat equation in θ.

Initial boundary value problems of the type (1.1) have been studied by several authors ([15], [16], [21], [27], [28], [32], etc.; see [25] for a review). Initial efforts to prove existence of solutions for this type of systems considered the heat flux in the form $q = -k\theta_x - \alpha k\theta_{xt}$, with $\alpha > 0$, instead of the classical Fourier law ($\alpha = 0$). This assumption introduces the additional term $-\alpha k\theta_{xxt}$ on the left hand side of the second equation in (1.1). Although this was done merely for mathematical reasons so that existence theorems could be proved ([15], [16], [21], [22]), it turns out that the second law of thermodynamics is not satisfied if $\alpha > 0$, as it can be easily verified by checking the Clausius-Duhem inequality for the entropy production. Therefore, the case $\alpha > 0$ has no physical meaning. The first results on existence of solutions for the case $\alpha = 0$ are due to Sprekels ([27]). However, he imposed very strong growth conditions on the free energy Ψ. In particular, those conditions excluded the physically relevant case in which Ψ is given in the Landau-Ginzburg form (1.2). Later on, Zheng ([32]) derived certain apriori estimates from which he concluded that, if the initial data is smooth enough, then any local solution of (1.1) with Ψ as in (1.2) can be extended globally in time. This result was later generalized by Sprekels and Zheng ([28]) to include more general free energy functionals. More recently, using a state-space approach ([25]) it was shown that system (1.1)-(1.2) has a local solution for a much broader set of initial data than the one considered in [28] and [32].

From a practical point of view it would be very important to find the values of all the parameters in (1.1)-(1.2) that "best fit" experimental data for a given material. This is called the parameter identification problem (ID problem in the sequel). Once this problem is solved, the next step is to determine how well this model can predict the dynamics of a given shape memory material which is subjected to certain external inputs. This is called the model validation problem. Although numerical experiments performed with system (1.1) have shown that physically reasonable results can be obtained for certain values of the parameters (see [4] and [19]), the ID problem still remains open.

In order to establish the convergence of computational algorithms for parameter identification, one needs to show first that the solutions depend continuously on the parameters that one wants to estimate. As we shall see in the following section, system (1.1)-(1.2) can be written as a semilinear Cauchy problem of the form $\dot{z}(t) = A(q)z(t)+F(q,t,z)$, $z(0) = z_0$, in an appropriate Hilbert space Z_q, where q is a vector of admissible parameters, $A(q)$ is a certain differential operator associated with the linear part of the partial differential equations in (1.1) and $F(q,t,z)$ corresponds to the nonlinear part of the system. In [26] it was shown that the nonlinear term $F(q,t,z)$ is locally Lipschitz continuous in the state variable z in the topology of the graph of $(-A(q))^{\delta}$, for any $\delta > \frac{3}{4}$. Although this result is necessary to show the continuous dependence of the solutions of (1.1) with respect to the parameter q, it is not sufficient. In fact, it turns out that a key step in achieving this result involves proving that if $\{q_n\}_{n=1}^{\infty}$ is a sequence of admissible parameters converging to q, then the associated analytic semigroups $T(t;q_n)$ converge strongly to $T(t;q)$ in the norm of the graph of $(-A(q))^\delta$. This is a much stronger result than the one obtained by using the well known Trotter-Kato Theorem (see [25], Theorem 4.1).

2. PRELIMINARIES AND STATE-SPACE FORMULATION

In the sequel, an *isomorphism* will be understood to denote a bounded invertible operator from a Banach space onto another.

Let X be a Banach space and X^* its topological dual. We denote with $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ the value of x^* at x. For each $x \in X$ we define the *duality set* $S(x) \doteq \{x^* \in X^* : \langle x^*, x\rangle =$
$\|x\|^2 = \|x^*\|^2\}$. The Hahn-Banach theorem implies that $S(x)$ is nonempty for every $x \in X$. If A is a linear operator in X with domain $D(A)$, we say that A is *dissipative* if for every $x \in D(A)$ there exists $x^* \in S(x)$ such that $\mathrm{Re}\langle Ax, x^*\rangle \leq 0$. We say that A is *strictly dissipative* if A is dissipative and the condition $\mathrm{Re}\langle Ax, x^*\rangle = 0$ for all $x^* \in S(x)$ implies that $x = 0$. If X is a Hilbert space then $S(x) = \{x\}$ and therefore A is dissipative iff $\mathrm{Re}\langle Ax, x\rangle \leq 0$ for every $x \in D(A)$. We say that the operator A is *maximal dissipative* if A is dissipative and it has no proper dissipative extension. We say that the operator A is *(maximal) accretive* if $-A$ is (maximal) dissipative. If the operator A is strictly dissipative and maximal dissipative, we will simply say that A is *strictly maximal dissipative.*

If A generates a strongly continuous semigroup $T(t)$ on X then the *type of* T is defined to be the real number $w_0(T) \doteq \inf_{t>0} \frac{1}{t} \log \|T(t)\|$. It can be shown that the type of a semigroup is either finite or equals $-\infty$. Moreover, $w_0(T) = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|$. Also, the semigroup $T(t)$ is of negative type iff $T(t)$ is exponentially stable, i.e., $w_0(T) < 0$ iff $\exists M \geq 1$, $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t > 0$ (see [1, pp 17-21]). If the semigroup $T(t)$ generated by A is analytic and $\sigma(A)$ denotes the spectrum of A, then $w_0(T) = \sup_{\lambda\in\sigma(A)} \mathrm{Re}\,\lambda$ provided that $\sigma(A) \neq \emptyset$ and $w_0(T) = -\infty$ if $\sigma(A) = \emptyset$ (see [1]).

Let us return now to our original problem (1.1)-(1.2). We define the function $L(x,t) \doteq \theta_\Gamma(t)\cos(2\pi x)$ and the transformation $\tilde{\theta}(x,t) = \theta(x,t) - L(x,t)$. We also define the *state space* $Z \doteq H_0^1(0,1) \cap H^2(0,1) \times L^2(0,1) \times L^2(0,1)$, $z \doteq \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z$ and the *admissible parameter set*

$$\mathcal{Q} \doteq \left\{ q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \;\mid\; q \in \mathrm{I\!R}^8_{>0} \right\}.$$

Next, we define in Z an inner product $\langle\cdot,\cdot\rangle_q$ depending on the parameter q as follows

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle_q \doteq \gamma \int_0^1 u''(x)\hat{u}''(x)\,dx + \rho \int_0^1 v(x)\hat{v}(x)\,dx + \frac{C_v}{k} \int_0^1 w(x)\hat{w}(x)\,dx$$

and we denote by Z_q the Hilbert space Z endowed with the inner product $\langle\cdot,\cdot\rangle_q$. The norm induced by $\langle\cdot,\cdot\rangle_q$ in Z_q will be denoted by $\|\cdot\|_q$. Note that these norms are all equivalent and, moreover, they are uniformly equivalent on compact subsets of $\mathcal{Q}$. Then the initial boundary value problem (1.1) with Ψ as in (1.2) can be formally written as an abstract semilinear Cauchy problem in Z_q as follows

$$\begin{cases} \dot{z}(t) = A(q)z(t) + F(q,t,z(t)), & 0 \le t \le T \\ z(0) = z_0, \end{cases} \tag{2.1}$$

where $z(t)(x) = \begin{pmatrix} u(x,t) \\ u_t(x,t) \\ \tilde{\theta}(x,t) \end{pmatrix}$,

$$D\left(A(q)\right) \doteq \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q \;\middle|\; \begin{array}{l} u \in H^4(0,1),\, u(0) = u(1) = 0 = u''(0) = u''(1), \\ v \in H_0^1(0,1) \cap H^2(0,1), \\ w \in H^2(0,1), \quad w'(0) = 0, \quad kw'(1) = -k_1 w(1) \end{array} \right\}, \tag{2.2}$$

and for $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D\left(A(q)\right)$,

$$A(q)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \doteq \begin{pmatrix} v \\ \beta v'' - \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -\frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{2.3}$$

The element z_0 is defined by

$$z_0(x) = \begin{pmatrix} u_0(x) \\ u_1(x) \\ \theta_0(x) - \theta_\Gamma(0)cos(2\pi x) \end{pmatrix}$$

and the nonlinear mapping $F(q,t,z) : \mathcal{Q} \times [0,T] \times Z_q \to Z_q$ is defined by

$$F(q,t,z) = F\left(q, t, \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) \doteq \begin{pmatrix} 0 \\ f_2(q,t,z) \\ f_3(q,t,z) \end{pmatrix}, \tag{2.4}$$

where

$$\begin{aligned} \rho f_2(q,t,z)(x) &= f(x,t) \\ &\quad + \frac{\partial}{\partial x}\left[2\alpha_2(w(x) + L(x,t) - \theta_1)u'(x) - 4\alpha_4 u'(x)^3 + 6\alpha_6 u'(x)^5\right], \\ C_v f_3(q,t,z)(x) &= g(x,t) + 2\alpha_2\left(w(x) + L(x,t)\right)u'(x)v'(x) \\ &\quad + \beta\rho v'(x)^2 - C_v\theta_\Gamma'(t)\cos(2\pi x) \\ &\quad - 4k\pi^2 L(x,t). \end{aligned}$$

The following results can be found in [25] and [26].

Theorem 2.1. ([25]) *Let $q \in \mathcal{Q}$ and the operator $A(q) : D(A(q)) \subset Z_q \to Z_q$ as defined by (2.2)-(2.3). Then*
i) *$A(q)$ is strictly maximal dissipative;*
ii) *The adjoint $A^*(q)$ is also strictly maximal dissipative and is given by $D(A^*(q)) = D(A(q))$, and for $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D(A^*(q))$*

$$A^*(q)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -v \\ \beta v'' + \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix} = \begin{pmatrix} 0 & -I & 0 \\ \frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix};$$

iii) *$0 \in \rho(A(q))$, the resolvent set of $A(q)$;*
iv) *The spectrum $\sigma(A(q))$ of $A(q)$ consists only of eigenvalues, $\sigma(A(q)) = \sigma_p(A(q)) = \{\lambda_n^{+,-}, \alpha_n\}_{n=1}^{\infty}$ where $\lambda_n^{+,-} = \sqrt{\mu_n}\left(-r(q) \pm \sqrt{r^2(q) - 1}\right)$, $\alpha_n = -\frac{k\tau_n^2}{C_v}$, with $\mu_n = \dfrac{\gamma n^4 \pi^4}{\rho}$, $r(q) = \dfrac{\beta\sqrt{\rho}}{2\sqrt{\gamma}}$ and $\{\tau_n\}_{n=1}^{\infty}$ are all the positive solutions of the equation $\tan\tau = \dfrac{k_1}{k\tau}$. The corresponding set of normalized eigenvectors in Z_q is given by*

$$\left\{ \begin{pmatrix} e_n \\ \lambda_n^+ e_n \\ 0 \end{pmatrix}, \begin{pmatrix} k_n e_n \\ k_n \lambda_n^- e_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix} \right\}_{n=1}^{\infty},$$

where $e_n(x) = \left(\dfrac{2}{\rho\left(\mu_n + |\lambda_n^+|^2\right)}\right)^{\frac{1}{2}} \sin(\pi n x)$, $\chi_n(x) = \left(\dfrac{k\tau_n}{C_v \int_0^{\tau_n} \cos^2(\xi)\, d\xi}\right)^{\frac{1}{2}} \cos(\tau_n x)$ *and* $k_n^2 = \frac{\mu_n + |\lambda_n^+|^2}{\mu_n + |\lambda_n^-|^2}$.
v) *The operator $A(q)$ generates an analytic semigroup $T(t;q)$ of negative type which satisfies $\|T(t;q)\|_{\mathcal{L}(Z_q)} \leq e^{-\omega(q)t}$, for $t \geq 0$, where $\omega(q)$ is given by*

$$\omega(q) = \begin{cases} \min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2}\right), & \textit{if } \beta^2\rho \leq 4\gamma \\ \min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2} - \frac{\pi^2}{2\sqrt{\rho}}\sqrt{\beta^2\rho - 4\gamma}\right), & \textit{if } \beta^2\rho > 4\gamma. \end{cases}$$

It will be useful to introduce some notation for certain interpolation spaces. If X is a Banach space and $p \geq 1$, $L_*^p(X)$ will denote the Banach space of all Bochner measurable mappings $u : [0,\infty) \to X$ such that $\|u\|_{L_*^p(X)}^p \doteq \int_0^\infty \|u(t)\|_X^p \frac{dt}{t} < \infty$. Let X_0, X_1 be two Banach spaces with X_0 continuously and densely embedded in X_1, $p \geq 1$ and $\theta \in (0,1)$. We shall denote by $(X_0, X_1)_{\theta,p}$ the space of averages (or "real" interpolation space)

$$(X_0, X_1)_{\theta,p} \doteq \left\{ x \in X_1 \;\middle|\; \begin{array}{l} \exists u_i : [0,\infty) \to X_i, i = 0,1, \quad t^{-\theta}u_0 \in L_*^p(X_0), \\ t^{1-\theta}u_1 \in L_*^p(X_1) \text{ and } x = u_0(t) + u_1(t) \text{ a.e.} \end{array} \right\}.$$

Endowed with the norm

$$\|x\|_{(X_0,X_1)_{\theta,p}} \doteq \inf\left\{ \|t^{-\theta}u_0\|_{L_*^p(X_0)} + \|t^{1-\theta}u_1\|_{L_*^p(X_1)} \;\middle|\; \begin{array}{l} t^{-\theta}u_0 \in L_*^p(X_0), \\ t^{1-\theta}u_1 \in L_*^p(X_1) \text{ and} \\ x = u_0(t) + u_1(t) \text{ a.e.} \end{array} \right\},$$

$(X_0, X_1)_{\theta,p}$ is a Banach space. In the particular case when $p = 2$ and X_0, X_1 are Hilbert spaces, we shall denote $(X_0, X_1)_{\theta,2} = [X_0, X_1]_\theta$.

Since $0 \in \rho(A(q))$ and $A(q)$ generates an analytic semigroup $T(t;q)$, the fractional δ-powers $(-A(q))^\delta$ of $-A(q)$ are well defined, closed, linear, invertible operators for any $\delta \geq 0$ (see [23, pp 69-75]). Moreover, $(-A(q))^{-\delta}$ has the representation

$$(-A(q))^{-\delta} = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} T(t;q)\, dt,$$

where the integral converges in the uniform operator topology for every $\delta > 0$. Since $A(q)$ is closed and $0 \in \rho(A(q))$, the operator $(-A(q))^\delta$ is also closed and invertible for each $\delta > 0$. Therefore, $D\left((-A(q))^\delta\right)$ endowed with the topology of the graph norm is a Hilbert space. Since $((-A(q))^\delta$ is boundedly invertible, the norm of the graph of $((-A(q))^\delta$ is equivalent to the norm $\|z\|_{q,\delta} \doteq \left\|(-A(q))^\delta z\right\|_q$. We shall denote by $Z_{q,\delta}$ the Hilbert space $D\left((-A(q))^\delta\right)$ endowed with the $\|\cdot\|_{q,\delta}$-norm.

Theorem 2.2. ([26]) *Let $q \in \mathcal{Q}$, $A(q) : D(A(q)) \subset Z_q \to Z_q$ as defined by (2.2)-(2.3), $0 < \delta < 1$ and $Z_{q,\delta}$ as defined above. Then*

- **i)** *$Z_{q,\delta} = [D(A(q)), Z_q]_{1-\delta}$, in the sense of an isomorphism;*
- **ii)** *The norms $\|z\|_{q,\delta}$, $\|z\|_{(D(A(q)),Z_q)_{1-\delta,2}}$ and $\|z\|_q + \left\|t^{1-\delta} A(q) T(t;q) z\right\|_{L^2_*(Z_q)}$ are all equivalent in $D\left((-A(q))^\delta\right)$.*

The next lemma shows some relations between the spaces $Z_{q,\delta}$ for different q's.

Lemma 2.3. ([26]) *Let $\delta \in (0,1)$. Then,*

- **i)** *For any pair q, $q^* \in \mathcal{Q}$ the spaces $Z_{q,\delta}$ and $Z_{q^*,\delta}$ are isomorphic.*
- **ii)** *Moreover, for any compact subset $\mathcal{Q}_C$ of $\mathcal{Q}$ the norms $\{\|\cdot\|_{q,\delta} : q \in \mathcal{Q}_C\}$ are uniformly equivalent, i.e., there exist positive constants m, M such that $m\|z\|_{q,\delta} \leq \|z\|_{q^*,\delta} \leq M\|z\|_{q,\delta}$ for every q, $q^* \in \mathcal{Q}_C$ and all $z \in D\left((-A(q))^\delta\right) \cap D\left((-A(q^*))^\delta\right)$.*

Consider the following standing hypotheses.

(H1) There exist functions K_f, $K_g \in L^2(0,1)$, $K_f(x) \geq 0$ a.e., $K_g(x) \geq 0$ a.e., such that

$$|f(x,t_1) - f(x,t_2)| \leq K_f(x)\,|t_1 - t_2| \quad \text{and} \quad |g(x,t_1) - g(x,t_2)| \leq K_g(x)\,|t_1 - t_2|$$

for a.e. $x \in (0,1)$ and all t_1, $t_2 \in [0,T]$.

(H2) $\theta_\Gamma \in H^1(0,T)$ and θ'_Γ is locally Lipschitz continuous in $(0,T)$.

Theorem 2.4. ([26]) *Let $q \in \mathcal{Q}$, $0 < \epsilon < \frac{1}{4}$ and assume that the hypotheses (H1) and (H2) hold. Then,*

- **i)** *for any bounded subset U of $[0,T] \times Z_{q,\frac{3}{4}+\epsilon}$ there exists a constant $L = L(q, U, \theta_\Gamma, f, g)$ such that*

$$\|F(q,t_1,z_1) - F(q,t_2,z_2)\|_q \leq L\left(|t_1 - t_2| + \|z_1 - z_2\|_{q,\frac{3}{4}+\epsilon}\right)$$

for all (t_1, z_1), $(t_2, z_2) \in U$, *i.e., the function* $F(q,t,z) : \mathcal{Q} \times [0,T] \times Z_{q,\frac{3}{4}+\epsilon} \to Z_q$ *is locally Lipschitz continuous in* t *and* z*. Moreover the constant* L *can be chosen independent of* q *on any compact subset of* $\mathcal{Q}$*;*

ii) *for any initial data* $z_0 \in D\left((-A(q))^{\frac{3}{4}+\epsilon}\right)$*, there exists* $t_1 = t_1(q, z_0) > 0$ *such that the initial value problem (2.1) has a unique strong solution* $z(t;q) \in C([0,t_1) : Z_q) \cap C^1((0,t_1) : Z_q)$*. Moreover* $\frac{d}{dt}z(t;q) \in C^{\frac{1}{4}-\epsilon}_{\mathrm{loc}}((0,t_1] : Z_q)$*, i.e.,* $\frac{d}{dt}z(t;q)$ *is locally Hölder continuous on* $(0,t_1]$ *with exponent* $\frac{1}{4}-\epsilon$*.*

Finally, we state the following theorem proved in [26], which states that for any compact subset $\mathcal{Q}_C$ of the admissible parameter set $\mathcal{Q}$, it is possible to find a nontrivial common interval of existence for all solutions $z(t,q)$, $q \in \mathcal{Q}_C$.

Theorem 2.5. ([26]) *Let* $\mathcal{Q}_C$ *be a compact subset of the admissible parameter set* $\mathcal{Q}$*,* $q_0 \in \mathcal{Q}_C$*,* $z_0 \in Z_{q_0,\delta}$*, where* $\frac{3}{4} < \delta < 1$*. Let* $\left[0, t^M(q)\right) = \left[0, t^M(q, z_0)\right)$ *denote the maximum interval of existence of the solution* $z(t;q)$ *with initial condition* $z(0;q) = z_0$*. Then*

$$t^M(\mathcal{Q}_C) \doteq \inf_{q \in \mathcal{Q}_C} t^M(q) > 0$$

3. CONTINUOUS DEPENDENCE ON THE MODEL PARAMETERS

In this section we show that the mapping $q \to z(\cdot\,; q)$ from the space of admissible parameters $\mathcal{Q}$ into the space of solutions is continuous. More precisely, we shall show that if $\{q_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{Q}$ converging to $q \in \mathcal{Q}$, then the sequence $\{z(t;q_n)\}_{n=1}^{\infty}$ converges to $z(t;q)$ in some appropriate sense.

Throughout this section, to simplify the notation we will denote with $A_n = A(q_n)$, $A = A(q)$, $T_n(t) = T(t;q_n)$, $T(t) = T(t;q)$, $z_n(t) = z(t;q_n)$ and $z(t) = z(t;q)$.

We shall need the following lemmas.

Lemma 3.1. *Let* $\{q_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{Q}$*,* $q_n \to q \in \mathcal{Q}$*, and let* A*,* A_n*,* T*,* T_n *be as above. Then*

$$\|A_n T_n(t) z - A T(t) z\|_q \to 0 \qquad \text{as } n \to \infty$$

for every $z \in Z_q$ *and* $t > 0$*.*

Proof. Let $z \in Z_q$. Since $T_n(t)$, $T(t)$ are analytic semigroups, $T_n(t)z$, $T(t)z$, are in $D(A_n)$, $D(A)$, respectively $\forall t > 0$. From Theorem 3.5 in [25] it follows that there exists an angle θ, $0 < \theta < \frac{\pi}{2}$, such that the angular sector

$$\Sigma_\theta = \{0\} \cup \{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \theta\} \subset \rho(A) \cap \bigcap_{n=1}^{\infty} \rho(A_n).$$

Now, let $\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} + \theta$ and let Γ be the path composed of the two rays $re^{-i\theta_1}$, $re^{i\theta_1}$, $0 \leq r < \infty$, Γ oriented so that $\mathrm{Im}(\lambda)$ increases along Γ. We have the following expresions (see [23])

$$AT(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; A)z\, d\lambda,$$

$$A_n T_n(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; A_n)z\, d\lambda,$$

for every $z \in Z_q$, $t > 0$, where $R(\lambda; A) = (\lambda I - A)^{-1}$, $R(\lambda; A_n) = (\lambda I - A_n)^{-1}$.

Then

$$AT(t)z - A_n T_n(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} \left(R(\lambda; A) - R(\lambda; A_n)\right) z\, d\lambda. \tag{3.1}$$

But

$$\begin{aligned} \|\lambda e^{\lambda t} \left(R(\lambda; A) - R(\lambda; A_n)\right) z\|_q &\leq |\lambda| e^{\mathrm{Re}(\lambda)t} \left(\frac{1}{|\lambda|} + \frac{C}{|\lambda|}\right) \|z\|_q \\ &\leq (1 + C) e^{\mathrm{Re}(\lambda)t} \|z\|_q \in L^1(\Gamma), \end{aligned}$$

where the constant C appears because of the uniform equivalence of the norms $\|\cdot\|_{q_n}$ and $\|\cdot\|_q$. Also, for any fixed $\lambda \in \Gamma$

$$\|\left(R(\lambda; A) - R(\lambda; A_n)\right) z\|_q \to 0 \qquad \text{as } n \to \infty.$$

In fact,

$$\begin{aligned} \|\left(R(\lambda; A) - R(\lambda; A_n)\right) z\|_q &= \|R(\lambda; A_n)\left[(\lambda I - A_n) R(\lambda; A) - I\right] z\|_q \\ &= \|R(\lambda; A_n)(A - A_n) R(\lambda; A) z\|_q \\ &\leq \|R(\lambda; A_n)\|_{\mathcal{L}(Z_q)} \|(A - A_n) R(\lambda; A) z\|_q \end{aligned}$$

which converges to zero as n goes to infinity by virtue of the uniform boundedness of $\|R(\lambda; A_n)\|_{\mathcal{L}(Z_q)}$ and the strong convergence of A_n to A (which follows immediately from the definition of A_n and A, and the convergence of q_n to q).

The lemma then follows from (3.1) and the Dominated Convergence Theorem. ■

Lemma 3.2. *Under the same hypotheses of Lemma 3.1*

$$\left\|(-A)^\delta (T(t) - T_n(t)) z\right\|_q \to 0 \qquad \textit{as } n \to \infty$$

for every $z \in Z_q$, $\delta \in [0, 1]$ and $t \geq 0$.

Remark. We note here that the assertion of Lemma 3.2 could be obtained immediately if $(-A)^\delta$ commuted with $T_n(t)$. However, this is not true since A_n does not commute with A, as it can be easily verified.

Proof of Lemma 3.2. It suffices to show the result for $\delta = 1$. We can write

$$\begin{aligned}\|A(T(t) - T_n(t))z\| &= \left\|[AT(t) - A_nT_n(t) + (I - AA_n^{-1})A_nT_n(t)]z\right\|_q \\ &\leq \|(AT(t) - A_nT_n(t))z\|_q + \left\|I - AA_n^{-1}\right\|_{\mathcal{L}(Z_q)} \|A_nT_n(t)z\|_q .\end{aligned}$$

As a consequence of Lemma 3.1 the first term on the right of the above inequality tends to zero as n goes to infinity and the sequence $\{\|A_nT_n(t)z\|_q\}_{n=1}^{\infty}$ is bounded. A straightforward calculation using the definition of $A(q)$ shows that for any pair of admissible parameters $q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma)$, $\tilde{q} = (\tilde{\rho}, \tilde{C}_v, \tilde{\beta}, \tilde{\alpha}_2, \tilde{\alpha}_4, \tilde{\alpha}_6, \tilde{\theta}_1, \tilde{\gamma})$ $\in \mathcal{Q}$ and any $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q$

$$A(\tilde{q})A^{-1}(q)z = \begin{pmatrix} u \\ \left(\tilde{\beta} - \beta\frac{\rho\tilde{\gamma}}{\tilde{\rho}\gamma}\right) u'' + \frac{\rho\tilde{\gamma}}{\tilde{\rho}\gamma} v \\ \left(\frac{C_v}{\tilde{C}_v}\right) w \end{pmatrix}, \tag{3.2}$$

from which it follows immediately that $\|I - AA_n^{-1}\|_{\mathcal{L}(Z_q)} \to 0$ as $n \to \infty$. The theorem then follows. ∎

Lemma 3.3. *Let $\mathcal{Q}_C$ be a compact subset of $\mathcal{Q}$. Then for any $\delta \in [0, 1]$ there exists a constant C depending only on δ and $\mathcal{Q}_C$ such that*

$$\left\|(-A(q_1))^{\delta}(-A(q_2))^{-\delta}\right\|_{\mathcal{L}(Z_{q_3})} \leq C$$

for every q_1, q_2, $q_3 \in \mathcal{Q}_C$.

Proof. Since the operator $A(q)$ is maximal dissipative (Theorem 2.1), the space $Z_{q,\delta}$ is isomorphic to the real interpolation space $[D(A(q)), Z_q]_{1-\delta}$, of order $1-\delta$ between Z_q and $D(A(q))$ (see [1]), i.e.

$$\left(D\left((-A(q))^{\delta}\right), \|\cdot\|_{q,\delta}\right) \cong [D(A(q)), Z_q]_{1-\delta}. \tag{3.3}$$

From (3.2) it follows that there exists a constant C depending only on $\mathcal{Q}_C$ such that $\|A(\tilde{q})A^{-1}(q)z\|_{\tilde{q}} \leq C\|z\|_{\tilde{q}}$ for every $q, \tilde{q} \in \mathcal{Q}_C$, $z \in Z_q$. Letting $\eta = A^{-1}(q)z$ we obtain

$$\|A(\tilde{q})\eta\|_{\tilde{q}} \leq C\|A(q)\eta\|_{\tilde{q}} \qquad \text{for all } q, \tilde{q} \in \mathcal{Q}_C,\ \eta \in D(A(q)). \tag{3.4}$$

Since the $\|\cdot\|_q$-norms are uniformly equivalent for $q \in \mathcal{Q}_C$, it follows from (3.4) and (3.3) that the norms $\|\cdot\|_{q,\delta}$ are also uniformly equivalent for $q \in \mathcal{Q}_C$. Thus, for any $q_1, q_2, q_3 \in \mathcal{Q}_C$

$$\begin{aligned}\|(-A(q_1))^{\delta}(-A(q_2))^{-\delta}z\|_{q_3} &\leq C_1\|(-A(q_1))^{\delta}(-A(q_2))^{-\delta}z\|_{q_1} \\ &= C_1\|(-A(q_2))^{-\delta}z\|_{q_1,\delta} \\ &\leq C_1\,C_2\|(-A(q_2))^{-\delta}z\|_{q_2,\delta} \\ &= C_1\,C_2\|z\|_{q_2} \\ &\leq C_1\,C_2\,C_3\|z\|_{q_3},\end{aligned}$$

where the constants C_i, $i = 1, 2, 3$, depend only on $\mathcal{Q}_C$ and δ. ■

Remark. Since $T_n(t)$ is an analytic semigroup of contractions, by a well known result on semigroup theory ([23]), for any $\delta \in (0, 1]$, there exists a constant C_δ independent of n such that

$$\left\|(-A_n)^\delta T_n(t)\right\|_{\mathcal{L}(Z_{q_n})} \leq \frac{C_\delta}{t^\delta |\cos\nu_n|}$$

where ν_n is any angle in $(\frac{\pi}{2}, \pi)$ for which

$$\rho(A_n) \supset \{0\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \nu_n\}.$$

As we mentioned in Lemma 3.1, in this case the angle ν_n above can be chosen independent of n. Hence, there exists a constant $\tilde{C}_\delta$ depending only on δ such that

$$\|(-A_n)^\delta T_n(t)\|_{\mathcal{L}(Z_{q_n})} \leq \frac{\tilde{C}_\delta}{t^\delta} \quad \forall n = 1, 2, \cdots .$$

Next, we state a lemma whose proof can be found in [14] (Lemma 7.1.1).

Lemma 3.4. *Suppose $L \geq 0$, $0 < \delta < 1$ and $a(t)$ is a nonnegative, locally integrable function on $0 \leq t \leq T$. Let $u(t)$ be a real valued function defined on $[0, T]$ satisfying*

$$u(t) \leq a(t) + L\int_0^t \frac{1}{(t-s)^\delta} u(s)\, ds$$

on this interval. Then, there exists a constant $K = K(\delta)$ such that

$$u(t) \leq a(t) + KL\int_0^t \frac{a(s)}{(t-s)^\delta}\, ds \qquad \text{for } 0 \leq t < T.$$

The following theorem will be essential for our main result.

Theorem 3.5. *Let $\delta \in \left(\frac{3}{4}, 1\right)$, $\{q_n\}_{n=1}^\infty \subset \mathcal{Q}$, $q_n \to q \in \mathcal{Q}$, and $z_n(t)$, $z(t)$ be the solutions of the IVP (2.1) with initial datum $z_0 \in D\left((-A)^\delta\right)$ corresponding to the parameters q_n and q, respectively, and let $[0, t_1)$ be the maximal interval of existence of $z(t)$. Then, for any $t_1' < t_1$ there exists a constant N_0 such that $z_n(t)$ exists on $[0, t_1']$ for every $n \geq N_0$ and a constant D such that*

$$\|z_n(t)\|_{q,\delta} \leq D, \quad \forall n \geq N_0,\ \forall t \in [0, t_1'].$$

Proof. Let $\delta \in \left(\frac{3}{4}, 1\right)$, $0 < t_1' < t_1$, and $t_1^n > 0$ be such that $z_n(t)$ exists on $[0, t_1^n)$ for each $n \in \mathbb{N}$. Then, for $t \in [0, \min\{t_1', t_1^n\})$

$$z(t) = T(t)z_0 + \int_0^t T(t-s)F(s, z(s))\, ds$$

$$z_n(t) = T_n(t)z_0 + \int_0^t T_n(t-s)F_n(s, z_n(s))\, ds,$$

which imply

$$
\begin{aligned}
\|z(t)-z_n(t)\|_{q,\delta} &= \|(-A)^{\delta}z(t)-(-A)^{\delta}z_n(t)\|_q \\
&\leq \left\|(-A)^{\delta}\left(T(t)-T_n(t)\right)z_0\right\|_q \\
&\quad + \left\|\int_0^t (-A)^{\delta}T(t-s)F(q,s,z(s))-(-A)^{\delta}T_n(t-s)F(q_n,s,z_n(s))\,ds\right\|_q \\
&\leq \left\|(-A)^{\delta}\left(T(t)-T_n(t)\right)z_0\right\|_q \\
&\quad + \left\|\int_o^t (-A)^{\delta}T(t-s)F(q,s,z(s))-(-A)^{\delta}T_n(t-s)F(q,s,z(s))\,ds\right\|_q \\
&\quad + \left\|\int_0^t (-A)^{\delta}T_n(t-s)\left[F(q,s,z(s))-F(q_n,s,z(s))\right]\,ds\right\|_q \\
&\quad + \left\|\int_0^t (-A)^{\delta}T_n(t-s)\left[F(q_n,s,z(s))-F(q_n,s,z_n(s))\right]\,ds\right\|_q \\
&\doteq I_1^n(t)+I_2^n(t)+I_3^n(t)+I_4^n(t).
\end{aligned}
$$

Note that, even when this last inequality is true on $[0,\min\{t_1',t_1^n\})$, $I_1^n(t)$, $I_2^n(t)$ and $I_3^n(t)$ are well defined on $[0,t_1']$.

We have the following estimates

$$
\begin{aligned}
I_3^n(t) &\leq \int_0^t \|(-A)^{\delta}T_n(t-s)\|_{\mathcal{L}(Z_q)}\|F(q,s,z(s))-F(q_n,s,z(s))\|_q\,ds \\
&\leq C_1\int_0^t \|(-A_n)^{\delta}T_n(t-s)\|_{\mathcal{L}(Z_{q_n})}\|F(q,s,z(s))-F(q_n,s,z(s))\|_q\,ds \\
&\leq C_1\int_0^t \frac{C_\delta}{(t-s)^{\delta}}\|F(q,s,z(s))-F(q_n,s,z(s))\|_q\,ds.
\end{aligned}
$$

The second and third inequality follow from Lemma 3.3 and the Remark preceding Lemma 3.4, respectively. Now, for any $s\in[0,t_1']$, $\|F(q,s,z(s))-F(q_n,s,z(s))\|_q\to 0$ as $n\to\infty$. Also, there exists a constant C_2 independent of n such that $\|F(q,s,z(s))-F(q_n,s,z(s))\|_q\leq C_2$ for every $s\in[0,t_1']$, which follows easily from the continuity of $z(s)$ and the definition of F. Therefore, $I_3^n(t)\to 0$ as $n\to\infty$ on $[0,t_1']$ by the Dominated Convergence Theorem and $I_3^n(t)\leq \dfrac{C_1C_2C_\delta}{1-\delta}t^{1-\delta}$, $\forall n\in\mathbb{N}$, $\forall t\in[0,t_1']$.

To estimate $I_2^n(t)$, observe that

$$
I_2^n(t)\leq \int_0^t \|(-A)^{\delta}\left(T(t-s)-T_n(t-s)\right)F(q,s,z(s))\|_q\,ds.
$$

Now, $\|F(q,s,z(s))\|_q$ is uniformly bounded on $[0,t_1']$, say $\|F(q,s,z(s))\|_q\leq C_3$,

$\forall t \in [0, t_1']$ and

$$
\begin{aligned}
\|(-A)^{\delta}(T(t-s)-&T_n(t-s))\|_{\mathcal{L}(Z_q)} \\
&\leq \|(-A)^{\delta}T(t-s)\|_{\mathcal{L}(Z_q)} + \|(-A)^{\delta}T_n(t-s)\|_{\mathcal{L}(Z_q)} \\
&\leq \|(-A)^{\delta}T(t-s)\|_{\mathcal{L}(Z_q)} + C\|(-A_n)^{\delta}T_n(t-s)\|_{\mathcal{L}(Z_{q_n})} \\
&\leq \frac{C_\delta}{(t-s)^{\delta}} + \frac{C\,C_\delta}{(t-s)^{\delta}} = \frac{C_4}{(t-s)^{\delta}}.
\end{aligned}
$$

On the other hand, for any $s \in [0, t_1']$ we have

$$
\|(-A)^{\delta}\left(T(t-s) - T_n(t-s)\right) F(q, s, z(s))\|_q \to 0 \quad \text{as} \quad n \to \infty
$$

by Lemma 3.2. Therefore $I_2^n(t) \to 0$ as $n \to \infty$ by the Dominated Convergence Theorem, and also $I_2^n(t) \leq \dfrac{C_3 C_4}{1-\delta} t^{1-\delta}$, $\forall n$, $\forall t \in [0, t_1']$.

In regard to $I_1^n(t)$ observe that

$$
\begin{aligned}
I_1^n(t) &= \left\|(-A)^{\delta}\left(T_n(t) - T(t)\right) z_0\right\|_q \\
&= \left\|(-A)^{\delta}(-A_n)^{-\delta}(-A_n)^{\delta}T_n(t)z_0 - (-A)^{\delta}T(t)z_0\right\|_q \\
&\leq C\left\|T_n(t)(-A_n)^{\delta}z_0\right\|_q + \left\|T(t)(-A)^{\delta}z_0\right\|_q \\
&\leq C\left\|T_n(t)\right\|_{\mathcal{L}(Z_q)} C\left\|(-A)^{\delta}z_0\right\|_q + \|T(t)\|_{\mathcal{L}(Z_q)}\left\|(-A)^{\delta}z_0\right\|_q \\
&\leq C_5\left\|(-A)^{\delta}z_0\right\|_q,
\end{aligned}
$$

where we have used that $z_0 \in D\left((-A)^{\delta}\right)$ and the semigroups are contractive. Also, by Lemma 3.2 $I_1^n(t) \to 0$ as $n \to \infty$.

Similarly,

$$
\begin{aligned}
I_4^n(t) &\leq \int_0^t \|(-A)^{\delta}T_n(t-s)\|_{\mathcal{L}(Z_q)}\|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q\, ds \\
&\leq C_6\int_0^t \frac{1}{(t-s)^{\delta}}\|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q\, ds.
\end{aligned}
$$

From the above estimates on $I_1^n(t)$, $I_2^n(t)$, $I_3^n(t)$ and $I_4^n(t)$, there follows

$$
\|z(t) - z_n(t)\|_{q,\delta} \leq \epsilon_n(t) + C_6\int_0^t \frac{1}{(t-s)^{\delta}}\|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q\, ds \quad (3.5)
$$

where, for all $t \in [0, t_1']$, $\epsilon_n(t) \doteq I_1^n(t) + I_2^n(t) + I_3^n(t)$ satisfies $0 \leq \epsilon_n(t) \leq C_7$ for all $n \in \mathbb{N}$ and $\epsilon_n(t) \to 0$ as $n \to \infty$. In particular, these conditions imply $\int_0^{t_1'} \epsilon_n(t)\, dt \to 0$ as $n \to \infty$.

Let $K = K(\delta)$ be as in Lemma 3.4 and define $\tilde{K} \doteq C_7 + C_6 C_7 K$ and $M \doteq \sup_{0\leq t\leq t_1'} \|z(t)\|_{q,\delta}$. From the continuity of $z(t)$ it follows that $M < \infty$. Let $n \in \mathbb{N}$. Since $z(0) = z_n(0) = z_0$, there exists $\delta_n > 0$ such that $\|z_n(t)\|_{q,\delta} \leq$

$M + 2\tilde{K}$ for all $t \in [0, \delta_n]$. Let L be a Lipschitz constant for F on the set $U \doteq [0, t_1'] \times \left\{ \|z\|_\delta \le M + 2\tilde{K} \right\}$, valid for q and all the q_n's. Then, from (3.5) and Lemma 3.4, we have

$$\|z_n(t) - z(t)\|_{q,\delta} \le f_n(t) \quad \text{on } 0 \le t \le \delta_n, \tag{3.6}$$

where $f_n(t) \doteq \epsilon_n(t) + C_6 LK \int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\, ds$, for $t \in [0, t_1']$.

Now,

$$\begin{aligned} \int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\, ds &\le \int_0^t \frac{C_7}{(t-s)^\delta}\, ds \\ &= C_7 \int_0^t \frac{1}{s^\delta}\, ds \\ &= \frac{C_7}{1-\delta} t^{1-\delta}. \end{aligned}$$

Choosing $\eta = \eta(L) > 0$ sufficiently small so that $t^{1-\delta} \le \dfrac{1-\delta}{2L}$ for every $t \in [0, \eta]$, it follows that

$$\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\, ds \le \frac{C_7}{2L} \quad \text{for every } t \in [0, \eta]. \tag{3.7}$$

On the other hand, if $\eta < t \le t_1'$

$$\begin{aligned} \int_0^t \frac{\epsilon_n(t)}{(t-s)^\delta}\, ds &= \int_0^t \frac{\epsilon_n(t-s)}{s^\delta}\, ds \\ &= \int_0^\eta \frac{\epsilon_n(t-s)}{s^\delta}\, ds + \int_\eta^t \frac{\epsilon_n(t-s)}{s^\delta}\, ds \\ &\le \frac{C_7}{2L} + \frac{1}{\eta^\delta} \int_0^t \epsilon_n(t-s)\, ds \\ &\le \frac{C_7}{2L} + \frac{1}{\eta^\delta} \int_0^{t_1'} \epsilon_n(s)\, ds. \end{aligned}$$

Hence, since $\int_0^{t_1'} \epsilon_n(s)\, ds \to 0$, there exists N_0 such that

$$\int_0^t \frac{\epsilon_n(t)}{(t-s)^\delta} \le \frac{C_7}{2L} + \frac{C_7}{2L} = \frac{C_7}{L} \quad \forall t \in [\eta, t_1'] \text{ and } n \ge N_0. \tag{3.8}$$

From (3.7) and (3.8) it follows that

$$f_n(t) \le C_7 + C_6 C_7 K \quad \forall t \in [0, t_1'] \text{ and } n \ge N_0. \tag{3.9}$$

Consequently, from (3.6) and (3.9)

$$\|z_n(t) - z(t)\|_{q,\delta} \le \tilde{K} \quad \forall n \ge N_0 \text{ and } t \in [0, \delta_n],$$

which implies

$$\|z_n(t)\|_{q,\delta} \le M + \tilde{K} \quad \forall n \ge N_0 \text{ and } t \in [0, \delta_n]. \tag{3.10}$$

Finally, let $n \ge N_0$ be fixed. We claim that $z_n(t)$ exists on $[0, t_1']$ and for $t \in [0, t_1']$, $\|z_n(t)\|_{q,\delta} < M + 2\tilde{K}$. In fact, suppose, on the contrary, that there exists $t^* \le t_1'$ such that $\|z_n(t^*)\|_{q,\delta} = M + 2\tilde{K}$ and $\|z_n(t)\|_{q,\delta} < M + 2\tilde{K}$ for $0 \le t < t^*$. Then, in (3.6), δ_n can be replaced by t^* and (3.10) follows with $\delta_n = t^*$, i.e. $\|z_n(t)\|_{q,\delta} \le M + \tilde{K}$ on $[0, t^*]$. This contradicts $\|z_n(t^*)\|_{q,\delta} = M + 2\tilde{K}$. The theorem then follows taking $D = M + 2\tilde{K}$. ∎

Theorem 3.6. *Under the same hypotheses of Theorem 3.5*

$$\|z_n(t) - z(t)\|_{q,\delta} \to 0, \quad \text{as } n \to \infty$$

for every $t \in [0, t_1)$.

Remark. If the initial data is smooth enough, then the results in [28] and [32] imply that $t_1 = \infty$ and therefore, this theorem ensures the $\| \cdot \|_{q,\delta}$-convergence of $z_n(t)$ to $z(t)$ on the whole interval $[0, \infty)$.

Proof of Theorem 3.6. Let $\delta \in \left(\frac{3}{4}, 1\right)$ and $t_1' < t_1$. By Theorem 3.5 there exist $N_0 \in \mathbb{N}$ and $D > 0$ such that $z_n(t)$ exists and $\|z_n(t)\|_{q,\delta} \le D$ on $[0, t_1']$ for every $n \ge N_0$. Following the steps of Theorem 3.5 we see that for every $t \in [0, t_1']$ and $n \ge N_0$

$$\begin{aligned}
\|z(t) - z_n(t)\|_{q,\delta} &\le \epsilon_n(t) + C_6 \int_0^t \frac{1}{(t-s)^\delta} \|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q \, ds \\
&\le \epsilon_n(t) + LC_6 \int_0^t \frac{1}{(t-s)^\delta} \|z(s) - z_n(s)\|_{q,\delta} \, ds
\end{aligned}$$

where $0 \le \epsilon_n(t) \le C_7$ and $\epsilon_n(t) \to 0$ as $n \to \infty$ for every $t \in [0, t_1']$. In the last inequality we have used the fact that F is locally Lipschitz continuous and $\|z_n(t)\|_{q,\delta} \le D$, $\forall n \ge N_0$, $\forall t \in [0, t_1']$.

Hence, by Lemma 3.4, there exists $K > 0$ such that

$$\|z(t) - z_n(t)\|_{q,\delta} \le \epsilon_n(t) + K \int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta} \, ds \longrightarrow 0 \quad \text{as } n \to \infty.$$

Since t_1' is arbitrary, the theorem follows. ∎

4. CONCLUSIONS

In this paper we have shown that the solutions of the IBVP (1.1), with free energy potential Ψ in the Landau-Ginzburg form (1.2), depend continuously on the parameters ρ, C_v, β, α_2, α_4, α_6, θ_1 and γ. In particular, we have shown that if $\{q_n = (\rho_n, C_{v,n}, \beta_n,\ \alpha_{2,n}, \alpha_{4,n}, \alpha_{6,n}, \theta_{1,n}, \gamma_n)\}_{n=1}^{\infty}$ is a sequence of admissible parameters converging to the admissible parameter q, then not only $z(t;q_n) \to z(t;q)$ in the norm of Z_q, but also in the stronger $\|\cdot\|_{q,\delta}$-norm ($\delta = \frac{3}{4} + \epsilon$). This constitutes an important step towards solving the parameter identifiability and the ID problems for system (1.1). These problems, to which we are already devoting efforts, involve also showing that the mapping $q \to z(\cdot\,;q)$ from the admissible parameter set $\mathcal{Q}$ into the space of solutions is locally one-to-one. Results on this issue will be published in a forthcoming article.

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