JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **212**, 292-315 (1997)

Identifiability of the Landau-Ginzburg Potential in a Mathematical Model of Shape Memory Alloys

PEDRO MORIN †

and

RUBEN D. SPIES † ‡

Instituto de Desarrollo Tecnológico para la Industria Química - INTEC
Programa Especial de Matemática Aplicada - PEMA
CONICET - Universidad Nacional del Litoral
3000 Santa Fe - Argentina

Abstract:The nonlinear partial differential equations considered here arise from the conservation laws of linear momentum and energy, and describe structural phase transitions in one-dimensional shape memory solids with non-convex Landau-Ginzburg free energy potentials. In this article the theories of analytic semigroups and real interpolation spaces for maximal accretive operators are used to show that the solutions of the model depend continuously on the admissible parameters. Also, we show that the non-physical parameters that define the free energy are identifiable from the model.

1. Introduction

In this article we consider the following initial-boundary value problem (IBVP).

$$
(1.1)\begin{cases}
\rho u_{tt}-\beta\rho u_{xxt}+\gamma u_{xxxx}=f(x,t)+\frac{\partial}{\partial x}\left[\frac{\partial}{\partial \epsilon}\Psi(u_x,u_{xx},\theta)\right], & x\in\Omega, 0\le t\le T,\\
C_v\theta_t-k\theta_{xx}=g(x,t)+2\alpha_2\theta u_x u_{xt}+\beta\rho u_{xt}^2, & x\in\Omega, 0\le t\le T,\\
u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), \qquad \theta(x,0)=\theta_0(x), & x\in\Omega,\\
u(0,t)=u(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0, & 0\le t\le T,\\
\theta_x(0,t)=0, \qquad k\theta_x(1,t)=k_1\left(\theta_\Gamma(t)-\theta(1,t)\right), & 0\le t\le T.
\end{cases}
$$

The partial differential equations in (1.1) reflect the conservation of linear momentum and energy in a one-dimensional shape memory body $\Omega=(0,1)$. The functions, variables and parameters involved in (1.1) have the following physical meaning: $u(x,t)$ = displacement; $\theta(x,t)$ = absolute temperature; ρ = mass density; k = thermal conductivity coefficient; C_v = specific heat; β = viscosity coefficient; $f(x,t)$ = distributed forces acting on the body (input); $g(x,t)$ = distributed heat sources (input); $u_0(x)$ = initial displacement; $u_1(x)$ = initial velocity; $\theta_0(x)$ = initial temperature; $\theta_\Gamma(t)$ = temperature of the surrounding medium (input); k_1 = positive constant, proportional to the rate of thermal exchange at the right boundary, and T is a prescribed final time. The function Ψ, which represents the free energy density of the system, is assumed to be a function of the linearized shear strain $\epsilon=u_x$, the spatial derivative of the strain $\epsilon_x=u_{xx}$

†The work of the authors was supported in part by CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas and UNL, Universidad Nacional del Litoral, Santa Fe, Argentina, through project CAI+D 94-0016-004-023.
‡This research was supported in part by the Air Force Office of Scientific Research under grants F49620-93-1-0280 and F49620-96-1-0329 while the author was a visiting scientist at the Air Force Center for Optimal Design and Control, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0531.

and the temperature θ, and is taken in the Landau-Ginzburg form

$$\Psi(\epsilon, \epsilon_x, \theta) = \Psi_0(\theta) + \alpha_2(\theta - \theta_1)\epsilon^2 - \alpha_4\epsilon^4 + \alpha_6\epsilon^6 + \frac{\gamma}{2}\epsilon_x^2,$$
$$\Psi_0(\theta) = -C_v\theta \log\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C, \tag{1.2}$$

where θ_1, θ_2 are two critical temperatures and α_2, α_4, α_6, γ are positive constants, all depending on the material being considered and $C \geq 0$ is a fixed reference energy level. Note that for values of θ close to θ_1 and ϵ_x fixed, the function $\Psi(\epsilon, \epsilon_x, \theta)$ is a nonconvex function of ϵ. This property is related to the hysteresis phenomenon which caracterizes this type of materials in the low and intermediate temperature ranges. The stress-strain relations are strongly temperature-dependent. The behavior is elastic, ideally-plastic at low temperatures, superelastic at intermediate temperatures and almost linearly elastic in the high temperature range. Shape memory and solid-solid phase transitions (martensitic transformations) are other peculiar characteristics of these materials whose dynamical behavior is formally described by system (1.1). For a detailed review of these and other properties and the derivations of the equations in (1.1) we refer the reader to [28] and the references therein.

The boundary conditions mean that the beam is simply supported at both ends, thermally insulated at the left end and, at the right end, the rate of thermal exchange with the surrounding medium is prescribed. The nonlinear coupled equations in (1.1) are sometimes referred to as the equations of thermo-visco-elasto-plasticity. In particular, the first equation in (1.1) can be regarded as a nonlinear beam equation in u, while the second is a nonlinear heat equation in θ.

Initial boundary value problems of the type (1.1) have been studied by several authors ([16], [17], [22], [26], [27], [30], etc.; see [28] for a review). The first results on existence of solutions for IBVP's like (1.1) are due to Sprekels ([30]). However, he imposed very strong growth conditions on the free energy Ψ. In particular, these conditions excluded the physically relevant case in which Ψ is given in the Landau-Ginzburg form (1.2). Later on, Songmu ([26]) derived certain a-priori estimates from which he concluded that, if the initial data is smooth enough, then any local solution of (1.1) with Ψ as in (1.2) can be extended globally in time. This result was later generalized by Songmu and Sprekels ([27]) to include more general free energy functionals. More recently, using a state-space approach ([28]) it was shown that the IBVP (1.1) can be written as a semilinear Cauchy problem of the form $\dot{z}(t) = A(q)z(t) + F(q, t, z)$, $z(0) = z_0$, in an appropriate Hilbert space Z_q, where q is a vector of admissible parameters, $A(q)$ is a certain differential operator associated with the linear part of the partial differential equations in (1.1) and $F(q, t, z)$ corresponds to the nonlinear part of the system. This approach provides a friendly framework for a suitable treatment of several problems associated to (1.1) such as existence, uniqueness, regularity and asymptotic behavior of solutions, as well as a powerful tool for numerical approximations, parameter estimation and control.

From a practical point of view it would be very important to find the values of all the parameters in (1.1)-(1.2) that "best fit" experimental data for a given material. This is called the parameter identification problem (ID problem in the sequel). Once this problem is solved, the next step is to determine how well this model can predict the dynamics of a given shape memory material which is subjected to certain external inputs. This is called the model validation problem. Although numerical experiments performed with system (1.1) have shown that physically reasonable results can be obtained for certain values of the parameters (see [5] and [20]), the ID problem still remains open.

In order to establish the convergence of computational algorithms for parameter identification, one needs to show first that the solutions depend continuously on the parameters that one wants to estimate. In [29] it was shown that the nonlinear term $F(q, t, z)$ is Lipschitz continuous in the state variable z in the topology of the graph of $(-A(q))^\delta$, for any $\delta > \frac{3}{4}$. Although this result is necessary to show the continuous dependence of the solutions of (1.1) with respect to the parameter q, it is not sufficient. In fact, it turns out that a key step in achieving this result involves proving that if $\{q_n\}_{n=1}^\infty$ is a sequence of admissible parameters converging to q, then the sequence of analytic semigroups $T(t; q_n)$ generated by $A(q_n)$, converges strongly to $T(t; q)$ in the norm of the graph of $(-A(q))^\delta$. This is a much stronger result than the one we can obtain by a straighforward application of the well known Trotter-Kato Theorem (see [28, Theorem 4.1]).

2. Preliminaries and State-Space Formulation

In the sequel, an *isomorphism* will be understood to denote a bounded invertible operator from a Banach space onto another.

Let X be a Banach space and X^* its topological dual. We denote with $\langle x^*\,,\,x\rangle$ or $\langle x\,,\,x^*\rangle$ the value of x^* at x. For each $x \in X$ we define the *duality set* $S(x) \doteq \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$. The Hahn-Banach theorem implies that $S(x)$ is nonempty for every $x \in X$. If A is a linear operator in X with domain $D(A)$, we say that A is *dissipative* if for every $x \in D(A)$ there exists $x^* \in S(x)$ such that $\mathrm{Re}\langle Ax, x^*\rangle \leq 0$. We say that A is *strictly dissipative* if A is dissipative and the condition $\mathrm{Re}\langle Ax, x^*\rangle = 0$ for all $x^* \in S(x)$ implies that $x = 0$. If X is a Hilbert space then $S(x) = \{x\}$ and therefore A is dissipative iff $\mathrm{Re}\langle Ax, x\rangle \leq 0$ for every $x \in D(A)$. We say that the operator A is *maximal dissipative* if A is dissipative and it has no proper dissipative extension. We say that the operator A is (*maximal*) *accretive* if $-A$ is (maximal) dissipative. If the operator A is strictly dissipative and maximal dissipative, we will simply say that A is *strictly maximal dissipative*.

If A generates a strongly continuous semigroup $T(t)$ on X then the *type of* T is defined to be the real number $w_0(T) \doteq \inf\limits_{t>0} \dfrac{1}{t} \log\|T(t)\|$. It can be shown that the type of a semigroup is either finite or equals $-\infty$. Moreover, $w_0(T) = \lim\limits_{t\to\infty} \dfrac{1}{t}\log\|T(t)\|$. Also, the semigroup $T(t)$ is of negative type iff $T(t)$ is exponentially stable, i.e., $w_0(T) < 0$ iff $\exists M \geq 1$, $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$ (see [2, pp 17-21]). If the semigroup $T(t)$ generated by A is analytic, then $w_0(T) = \sup\limits_{\lambda\in\sigma(A)} \mathrm{Re}\,\lambda$ provided that $\sigma(A) \neq \emptyset$ and, by definition, $w_0(T) = -\infty$ if $\sigma(A) = \emptyset$ (see [2]).

Let us now return to our original problem (1.1)-(1.2). We define the function $L(x,t) \doteq \theta_\Gamma(t)\cos(2\pi x)$ and the transformation $\tilde\theta(x,t) = \theta(x,t) - L(x,t)$. We also define the *admissible parameter set* $\mathcal{Q} \doteq \{q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \mid q \in \mathbb{R}^8_{>0}\}$, and the *state space* Z_q as the Hilbert space $H^1_0(0,1) \cap H^2(0,1) \times L^2(0,1) \times L^2(0,1)$ endowed with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat u \\ \hat v \\ \hat w \end{pmatrix} \right\rangle_q \doteq \gamma \int_0^1 u''(x)\overline{\hat u''(x)}\,dx + \rho\int_0^1 v(x)\overline{\hat v(x)}\,dx + \frac{C_v}{k}\int_0^1 w(x)\overline{\hat w(x)}\,dx.$$

The norm induced by $\langle\cdot,\cdot\rangle_q$ in Z_q will be denoted by $\|\cdot\|_q$. Note that these norms are all equivalent and, moreover, they are uniformly equivalent on compact subsets of $\mathcal{Q}$. Then the initial-boundary value problem (1.1) with Ψ as in (1.2) can be formally written as an abstract semilinear Cauchy problem in Z_q as follows:

$$\begin{cases} \frac{d}{dt} z(t) = A(q)z(t) + F(q,t,z(t)), \qquad 0 \leq t \leq T \\ \\ z(0) = z_0, \end{cases} \tag{2.1}$$

where $z(t)(x) = \begin{pmatrix} u(x,t) \\ u_t(x,t) \\ \tilde\theta(x,t) \end{pmatrix}$,

$$D(A(q)) \doteq \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q \;\middle|\; \begin{array}{l} u \in H^4(0,1),\ u(0) = u(1) = 0 = u''(0) = u''(1), \\ v \in H^1_0(0,1) \cap H^2(0,1), \\ w \in H^2(0,1), \quad w'(0) = 0, \quad kw'(1) = -k_1 w(1) \end{array} \right\}, \tag{2.2}$$

and for $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D(A(q))$,

$$A(q)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \doteq \begin{pmatrix} v \\ \beta v'' - \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -\frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{2.3}$$

The element z_0 is defined by

$$z_0(x) = \begin{pmatrix} u_0(x) \\ u_1(x) \\ \theta_0(x) - \theta_\Gamma(0)\cos(2\pi x) \end{pmatrix}$$

and the nonlinear mapping $F(q,t,z) : \mathcal{Q} \times [0,T] \times Z_q \to Z_q$ is defined by

$$F(q,t,z) = F\left(q, t, \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) \doteq \begin{pmatrix} 0 \\ f_2(q,t,z) \\ f_3(q,t,z) \end{pmatrix}, \tag{2.4}$$

where

$$\rho f_2(q,t,z)(x) = f(x,t) + \frac{\partial}{\partial x}\left[2\alpha_2(w(x)+L(x,t)-\theta_1)u'(x) - 4\alpha_4 u'(x)^3 + 6\alpha_6 u'(x)^5\right],$$
$$C_v f_3(q,t,z)(x) = g(x,t) + 2\alpha_2\left(w(x)+L(x,t)\right)u'(x)v'(x) + \beta\rho v'(x)^2 - C_v\theta_\Gamma'(t)\cos(2\pi x) - 4k\pi^2 L(x,t).$$

The following results can be found in [28] and [29].

Theorem 2.1. ([28]) *Let $q \in \mathcal{Q}$ and the operator $A(q) : D(A(q)) \subset Z_q \to Z_q$ as defined by (2.2)-(2.3). Then*
i) *$A(q)$ is strictly maximal dissipative;*
ii) *The adjoint $A^*(q)$ is also strictly maximal dissipative and is given by $D(A^*(q)) = D(A(q))$,*

$$A^*(q)\begin{pmatrix} u \\ v \\ w\end{pmatrix} = \begin{pmatrix} -v \\ \beta v'' + \frac{\gamma}{\rho}u'''' \\ \frac{k}{C_v}w''\end{pmatrix} = \begin{pmatrix} 0 & -I & 0 \\ \frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2}\end{pmatrix}\begin{pmatrix} u \\ v \\ w\end{pmatrix}, \quad \begin{pmatrix} u \\ v \\ w\end{pmatrix} \in D(A^*(q));$$

iii) *$0 \in \rho(A(q))$, the resolvent set of $A(q)$;*
iv) *The spectrum $\sigma(A(q))$ of $A(q)$ consists only of eigenvalues, $\sigma(A(q)) = \sigma_p(A(q)) = \{\lambda_n^{+,-},\, \alpha_n\}_{n=1}^{\infty}$ where $\lambda_n^{+,-} = \sqrt{\mu_n}\left(-r(q) \pm \sqrt{r^2(q)-1}\right)$, $\alpha_n = -\frac{k\tau_n^2}{C_v}$, with $\mu_n = \dfrac{\gamma n^4\pi^4}{\rho}$, $r(q) = \dfrac{\beta\sqrt{\rho}}{2\sqrt{\gamma}}$ and $\{\tau_n\}_{n=1}^{\infty}$ are all the positive solutions of the equation $\tan\tau = \dfrac{k_1}{k\tau}$. The corresponding set of normalized eigenvectors in Z_q is given by*

$$\left\{\begin{pmatrix} e_n \\ \lambda_n^+ e_n \\ 0\end{pmatrix}, \begin{pmatrix} k_n e_n \\ k_n\lambda_n^- e_n \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \chi_n\end{pmatrix}\right\}_{n=1}^{\infty},$$

where $e_n(x) = \left(\dfrac{2}{\rho\left(\mu_n + |\lambda_n^+|^2\right)}\right)^{1/2}\sin(\pi n x)$, $\chi_n(x) = \left(\dfrac{k\tau_n}{C_v\int_0^{\tau_n}\cos^2(\xi)\,d\xi}\right)^{1/2}\cos(\tau_n x)$ *and* $k_n^2 = \frac{\mu_n + |\lambda_n^+|^2}{\mu_n + |\lambda_n^-|^2}$.
v) *The operator $A(q)$ generates an analytic semigroup $T(t;q)$ of negative type which satisfies $\|T(t;q)\|_{\mathcal{L}(Z_q)} \le e^{-\omega(q)t}$, for $t \ge 0$, where $\omega(q)$, the type of T, is given by*

$$\omega(q) = \begin{cases} \min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2}\right), & \text{if } \beta^2\rho \le 4\gamma \\ \min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2} - \frac{\pi^2}{2\sqrt{\rho}}\sqrt{\beta^2\rho - 4\gamma}\right), & \text{if } \beta^2\rho > 4\gamma. \end{cases}$$

We shall need some notation for certain interpolation spaces. If X is a Banach space and $p \ge 1$, $L_*^p(X)$ will denote the Banach space of all Bochner measurable mappings $u : [0,\infty) \to X$ such that $\|u\|_{L_*^p(X)}^p \doteq \int_0^\infty \|u(t)\|_X^p \frac{dt}{t} < \infty$. If X_0 and X_1 are two Banach spaces with X_0 continuously and densely embedded in X_1, $p \ge 1$ and $\theta \in (0,1)$, we denote by $(X_0, X_1)_{\theta,p}$ the space of averages (or "real" interpolation space)

$$(X_0,X_1)_{\theta,p} \doteq \left\{ x \in X_1 \;\middle|\; \begin{array}{l} \exists u_i : [0,\infty) \to X_i, i = 0,1, \quad t^{-\theta}u_0 \in L_*^p(X_0), \\ t^{1-\theta}u_1 \in L_*^p(X_1) \text{ and } x = u_0(t) + u_1(t) \text{ a.e.} \end{array}\right\}.$$

Endowed with the norm

$$\|x\|_{(X_0,X_1)_{\theta,p}} \doteq \inf\left\{ \|t^{-\theta}u_0\|_{L_*^p(X_0)} + \|t^{1-\theta}u_1\|_{L_*^p(X_1)} \;\middle|\; \begin{array}{l} t^{-\theta}u_0 \in L_*^p(X_0), \\ t^{1-\theta}u_1 \in L_*^p(X_1) \text{ and} \\ x = u_0(t) + u_1(t) \text{ a.e.} \end{array}\right\},$$

$(X_0,X_1)_{\theta,p}$ is a Banach space. In the particular case when $p = 2$ and X_0, X_1 are Hilbert spaces, we denote $(X_0,X_1)_{\theta,2} = [X_0,X_1]_\theta$.

Since $0 \in \rho\left(A(q)\right)$ and $A(q)$ generates an analytic semigroup $T(t;q)$, the fractional δ-powers $\left(-A(q)\right)^{\delta}$ of $-A(q)$ are well defined, closed, linear, invertible operators for any $\delta \geq 0$ (see [24, pp 69-75]). Moreover, $\left(-A(q)\right)^{-\delta}$ has the representation

$$\left(-A(q)\right)^{-\delta} = \frac{1}{\Gamma(\delta)} \int_0^{\infty} t^{\delta-1} T(t;q)\, dt,$$

where the integral converges in the uniform operator topology for every $\delta > 0$. Also, the domain $D\left(\left(-A(q)\right)^{\delta}\right)$ endowed with the topology of the norm of the graph of $\left(-A(q)\right)^{\delta}$ is a Hilbert space. Since $(\left(-A(q)\right)^{\delta}$ is boundedly invertible this norm is equivalent to the norm $\|z\|_{q,\delta} \doteq \left\|(-A(q))^{\delta} z\right\|_q$. We shall denote by $Z_{q,\delta}$ the Hilbert space $D\left((-A(q))^{\delta}\right)$ endowed with the $\|\cdot\|_{q,\delta}$-norm.

Next, we state a few results which will be needed throughout the rest of this article. Their proofs can be found in [29].

Theorem 2.2. ([29]) *Let $q \in \mathcal{Q}$, $A(q) : D\left(A(q)\right) \subset Z_q \to Z_q$ as defined by (2.2)-(2.3), $0 < \delta < 1$ and $Z_{q,\delta}$ as above. Then $Z_{q,\delta} = \left[D\left(A(q)\right), Z_q\right]_{1-\delta}$, in the sense of an isomorphism.*

The next lemma shows some relations between the spaces $Z_{q,\delta}$ for different q's.

Lemma 2.3. ([29]) *Let $\delta \in (0,1)$. Then,*

- **i)** *For any pair q, $q^* \in \mathcal{Q}$ the spaces $Z_{q,\delta}$ and $Z_{q^*,\delta}$ are isomorphic.*
- **ii)** *Moreover, for any compact subset $\mathcal{Q}_C$ of $\mathcal{Q}$ the norms $\{\|\cdot\|_{q,\delta} : q \in \mathcal{Q}_C\}$ are uniformly equivalent, i.e., there exist constants $m > 0$, $M > 0$, such that $m\|z\|_{q,\delta} \leq \|z\|_{q^*,\delta} \leq M\|z\|_{q,\delta}$ for every pair q, $q^* \in \mathcal{Q}_C$ and all $z \in D\left((-A(q))^{\delta}\right) \cap D\left((-A(q^*))^{\delta}\right)$.*

Consider the following standing hypotheses.

- **(H1)** There exist functions $K_f, K_g \in L^2(0,1)$, $K_f(x) \geq 0$ a.e., $K_g(x) \geq 0$ a.e., such that $|f(x,t_1) - f(x,t_2)| \leq K_f(x)\,|t_1 - t_2|$ and $|g(x,t_1) - g(x,t_2)| \leq K_g(x)\,|t_1 - t_2|$ for a.e. $x \in (0,1)$ and all t_1, $t_2 \in [0,T]$.
- **(H2)** $\theta_\Gamma \in H^1(0,T)$ and θ_Γ' is locally Lipschitz continuous in $(0,T)$.

Theorem 2.4. ([29]) *Let $q \in \mathcal{Q}$, $0 < \epsilon < \frac{1}{4}$ and assume that the hypotheses (H1) and (H2) hold. Then,*

- **i)** *for any bounded subset U of $[0,T] \times Z_{q,\frac{3}{4}+\epsilon}$ there exists a constant $L = L(q, U, \theta_\Gamma, f, g)$ such that*

$$\|F(q,t_1,z_1) - F(q,t_2,z_2)\|_q \leq L\left(|t_1 - t_2| + \|z_1 - z_2\|_{q,\frac{3}{4}+\epsilon}\right)$$

  for all (t_1,z_1), $(t_2,z_2) \in U$, i.e., the function $F(q,t,z) : \mathcal{Q} \times [0,T] \times Z_{q,\frac{3}{4}+\epsilon} \to Z_q$ is locally Lipschitz continuous in t and z. Moreover, the constant L can be chosen independent of q on any compact subset of $\mathcal{Q}$;
- **ii)** *for any initial data $z_0 \in D\left((-A(q))^{\frac{3}{4}+\epsilon}\right)$, there exists $t_1 = t_1(q,z_0) > 0$ such that the initial value problem (2.1) has a unique classical solution $z(t;q) \in C\left([0,t_1) : Z_q\right) \cap C^1\left((0,t_1) : Z_q\right)$. Moreover $\frac{d}{dt}z(t;q) \in C_{\mathrm{loc}}^{\frac{1}{4}-\epsilon}\left((0,t_1] : Z_q\right)$, i.e., $\frac{d}{dt}z(t;q)$ is locally Hölder continuous on $(0,t_1]$ with exponent $\frac{1}{4} - \epsilon$.*

Finally, the following theorem says that for any compact subset $\mathcal{Q}_C$ of the admissible parameter set $\mathcal{Q}$, it is possible to find a nontrivial common interval of existence for all solutions $z(t;q)$, $q \in \mathcal{Q}_C$.

Theorem 2.5. ([29]) *Let $\mathcal{Q}_C$ be a compact subset of the admissible parameter set $\mathcal{Q}$, $q_0 \in \mathcal{Q}_C$, $z_0 \in Z_{q_0,\delta}$, where $\frac{3}{4} < \delta < 1$. Let $\left[0, t^M(q)\right) = \left[0, t^M(q,z_0)\right)$ denote the maximum interval of existence of the solution $z(t;q)$ with initial condition $z(0;q) = z_0$. Then*

$$t^M(\mathcal{Q}_C) \doteq \inf_{q \in \mathcal{Q}_C} t^M(q) > 0$$

3. Continuous Dependence on the Model Parameters

In this section we show that the mapping $q \to z(\cdot\,; q)$ from the space of admissible parameters $\mathcal{Q}$ into the space of solutions is continuous. More precisely, we show that if $\{q_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{Q}$ converging to $q \in \mathcal{Q}$, then the sequence $\{z(t; q_n)\}_{n=1}^{\infty}$ converges to $z(t; q)$ in some appropriate sense.

Throughout this section, to simplify the notation we shall denote with $A_n = A(q_n)$, $A = A(q)$, $T_n(t) = T(t; q_n)$, $T(t) = T(t; q)$, $z_n(t) = z(t; q_n)$ and $z(t) = z(t; q)$.

We shall need the following lemmas.

Lemma 3.1. *Let $\{q_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{Q}$, $q_n \to q \in \mathcal{Q}$, and let A, A_n, T, T_n be as above. Then*

$$\|A_n T_n(t)z - AT(t)z\|_q \to 0 \qquad \text{as } n \to \infty$$

for every $z \in Z_q$ and $t > 0$.

Proof. Let $z \in Z_q$. Since $T_n(t)$, $T(t)$ are analytic semigroups, $T_n(t)z$, $T(t)z$, are in $D(A_n)$, $D(A)$, respectively, $\forall t > 0$. From Theorem 3.5 in [28] it follows that there exists an angle θ, $0 < \theta < \frac{\pi}{2}$, such that

$$\Sigma_\theta \doteq \{0\} \cup \{\lambda \in \mathbb{C} : |\arg\, \lambda| < \frac{\pi}{2} + \theta\} \subset \rho(A) \cap \bigcap_{n=1}^{\infty} \rho(A_n).$$

Now, let $\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} + \theta$ and let Γ be the path composed of the two rays $re^{-i\theta_1}$, $re^{i\theta_1}$, $0 \le r < \infty$, Γ oriented so that $\mathrm{Im}(\lambda)$ increases along Γ. We have the following expresions (see [24])

$$AT(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; A)z\, d\lambda, \qquad A_nT_n(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; A_n)z\, d\lambda,$$

for every $z \in Z_q$, $t > 0$, where $R(\lambda; A) = (\lambda I - A)^{-1}$, $R(\lambda; A_n) = (\lambda I - A_n)^{-1}$.

Then

$$AT(t)z - A_nT_n(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} \left(R(\lambda; A) - R(\lambda; A_n)\right) z\, d\lambda. \tag{3.1}$$

By the Hille-Yosida theorem ([24])

$$\begin{aligned} \|\lambda e^{\lambda t} \left(R(\lambda; A) - R(\lambda; A_n)\right) z\|_q &\le |\lambda| e^{\mathrm{Re}(\lambda)t} \left(\frac{1}{|\lambda|} + \frac{C}{|\lambda|}\right) \|z\|_q \\ &\le (1 + C)e^{\mathrm{Re}(\lambda)t}\|z\|_q \in L^1(\Gamma), \end{aligned}$$

where the constant C appears because of the uniform equivalence of the norms $\|\cdot\|_{q_n}$ and $\|\cdot\|_q$. Also for any fixed $\lambda \in \Gamma$, $\|\left(R(\lambda; A) - R(\lambda; A_n)\right) z\|_q \to 0$ as $n \to \infty$. In fact,

$$\begin{aligned} \|\left(R(\lambda; A) - R(\lambda; A_n)\right) z\|_q &= \|R(\lambda; A_n)\left[(\lambda I - A_n)R(\lambda; A) - I\right] z\|_q \\ &= \|R(\lambda; A_n)(A - A_n)R(\lambda; A)z\|_q \\ &\le \|R(\lambda; A_n)\|_{\mathcal{L}(Z_q)}\|(A - A_n)R(\lambda; A)z\|_q \end{aligned}$$

which converges to zero as n goes to infinity by virtue of the uniform boundedness of $\|R(\lambda; A_n)\|_{\mathcal{L}(Z_q)}$ and the strong convergence of A_n to A (which follows immediately from the definition of A_n and A, and the convergence of q_n to q).

The lemma then follows from (3.1) and the Dominated Convergence Theorem.

■

Lemma 3.2. *Under the same hypotheses of Lemma 3.1*

$$\left\|(-A)^\delta (T(t) - T_n(t))z\right\|_q \to 0 \qquad \text{as } n \to \infty$$

for every $z \in Z_q$, $\delta \in [0, 1]$ and $t \ge 0$.

Remark. Note that the assertion of Lemma 3.2 could be easily obtained if $(-A)^\delta$ commuted with $T_n(t)$. However, this is not true since A_n does not commute with A, as it can be easily verified.

Proof of Lemma 3.2. It suffices to show the result for $\delta = 1$. We write

$$\begin{aligned}\|A(T(t)-T_n(t))z\| &= \left\|[AT(t)-A_nT_n(t)+(I-AA_n^{-1})A_nT_n(t)]z\right\|_q \\ &\leq \|(AT(t)-A_nT_n(t))z\|_q + \left\|I-AA_n^{-1}\right\|_{\mathcal{L}(Z_q)} \|A_nT_n(t)z\|_q .\end{aligned}$$

As a consequence of Lemma 3.1 the first term on the right of the above inequality tends to zero as n goes to infinity and the sequence $\{\|A_nT_n(t)z\|_q\}_{n=1}^{\infty}$ is bounded. A straightforward calculation using the definition of $A(q)$ shows that for any pair of admissible parameters $q, \tilde{q} \in \mathcal{Q}$, $q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma)$, $\tilde{q} = (\tilde{\rho}, \tilde{C}_v, \tilde{\beta}, \tilde{\alpha}_2, \tilde{\alpha}_4, \tilde{\alpha}_6, \tilde{\theta}_1, \tilde{\gamma})$ and any $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q$

$$A(\tilde{q})A^{-1}(q)z = \begin{pmatrix} u \\ \left(\tilde{\beta} - \beta\frac{\rho\tilde{\gamma}}{\tilde{\rho}\gamma}\right)u'' + \frac{\rho\tilde{\gamma}}{\tilde{\rho}\gamma}v \\ \left(\frac{C_v}{\tilde{C}_v}\right)w \end{pmatrix}, \tag{3.2}$$

from which it follows immediately that $\|I - AA_n^{-1}\|_{\mathcal{L}(Z_q)} \to 0$ as $n \to \infty$. The theorem then follows.

■

Lemma 3.3. *Let $\mathcal{Q}_C$ be a compact subset of $\mathcal{Q}$. Then for any $\delta \in [0,1]$ there exists a constant C depending only on δ and $\mathcal{Q}_C$ such that*

$$\left\|(-A(q_1))^{\delta}(-A(q_2))^{-\delta}\right\|_{\mathcal{L}(Z_{q_3})} \leq C$$

for every q_1, q_2, $q_3 \in \mathcal{Q}_C$.

Proof. From (3.2) it follows that there exists a constant M depending only on $\mathcal{Q}_C$ such that $\|A(\tilde{q})A^{-1}(q)z\|_{\tilde{q}} \leq M\|z\|_{\tilde{q}}$ for every $q, \tilde{q} \in \mathcal{Q}_C$, $z \in Z_q$. Letting $\eta = A^{-1}(q)z$ we obtain

$$\|A(\tilde{q})\eta\|_{\tilde{q}} \leq M\|A(q)\eta\|_{\tilde{q}} \qquad \text{for all } q, \tilde{q} \in \mathcal{Q}_C,\ \eta \in D(A(q)). \tag{3.3}$$

Since the $\|\cdot\|_q$-norms are uniformly equivalent for $q \in \mathcal{Q}_C$, it follows from (3.3) and Theorem 2.2 that the norms $\|\cdot\|_{q,\delta}$ are also uniformly equivalent for $q \in \mathcal{Q}_C$. Thus, for any $q_1, q_2, q_3 \in \mathcal{Q}_C$

$$\begin{aligned}\|(-A(q_1))^{\delta}(-A(q_2))^{-\delta}z\|_{q_3} &\leq C_1\|(-A(q_1))^{\delta}(-A(q_2))^{-\delta}z\|_{q_1} \\ &= C_1\|(-A(q_2))^{-\delta}z\|_{q_1,\delta} \\ &\leq C_1\,C_2\|(-A(q_2))^{-\delta}z\|_{q_2,\delta} \\ &= C_1\,C_2\|z\|_{q_2} \\ &\leq C_1\,C_2\,C_3\|z\|_{q_3} \\ &\doteq C\|z\|_{q_3},\end{aligned}$$

where the constants C_i, $i = 1, 2, 3$, depend only on $\mathcal{Q}_C$ and δ.

■

Remark. Since $T_n(t)$ is an analytic semigroup of contractions, by a well known result on semigroup theory ([24]), there exists a constant $\tilde{C}_\delta$ independent of n such that

$$\left\|(-A_n)^{\delta}T_n(t)\right\|_{\mathcal{L}(Z_{q_n})} \leq \frac{\tilde{C}_\delta}{t^{\delta}|\cos\nu_n|}$$

where ν_n is any angle in $(\frac{\pi}{2}, \pi)$ for which

$$\rho(A_n) \supset \{0\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \nu_n\}.$$

As we mentioned in the proof of Lemma 3.1, the angle ν_n above can be chosen independent of n. Hence, there exists a constant C_δ depending only on δ such that

$$\|(-A_n)^{\delta}T_n(t)\|_{\mathcal{L}(Z_{q_n})} \leq \frac{C_\delta}{t^{\delta}} \quad \forall n = 1, 2, \cdots .$$

Next, we state a lemma whose proof can be found in [15, Lemma 7.1.1].

Lemma 3.4. *Suppose $L \geq 0$, $0 < \delta < 1$ and $a(t)$ is a nonnegative function, locally integrable on $0 \leq t \leq T$. Let $u(t)$ be a real-valued function defined on $[0,T]$ satisfying*

$$u(t) \leq a(t) + L \int_0^t \frac{1}{(t-s)^\delta} u(s)\, ds$$

on this interval. Then, there exists a constant K depending only on δ such that

$$u(t) \leq a(t) + KL \int_0^t \frac{a(s)}{(t-s)^\delta}\, ds \qquad \text{for } 0 \leq t < T.$$

The following theorem will be essential for the main result of this section.

Theorem 3.5. *Let $\delta \in \left(\frac{3}{4}, 1\right)$, $\{q_n\}_{n=1}^\infty \subset \mathcal{Q}$, $q_n \to q \in \mathcal{Q}$ and $z_n(t)$, $z(t)$ be the solutions corresponding to the parameters q_n and q, respectively, of the IVP (2.1) with initial data $z_0 \in D\left((-A)^\delta\right)$, and let $[0, t_1)$ be the maximal interval of existence of $z(t)$. Then, for any $t_1' < t_1$ there are constants N_0, $D > 0$ such that $z_n(t)$ exists on $[0, t_1']$ for every $n \geq N_0$ and*

$$\|z_n(t)\|_{q,\delta} \leq D, \qquad \forall n \geq N_0,\ \forall t \in [0, t_1'].$$

Proof. Let $\delta \in \left(\frac{3}{4}, 1\right)$, $0 < t_1' < t_1$, and $t_1^n > 0$ be such that $z_n(t)$ exists on $[0, t_1^n)$ for each $n \in \mathbb{N}$. Then, for $t \in [0, \min\{t_1', t_1^n\})$,

$$z(t) = T(t) z_0 + \int_0^t T(t-s) F(q, s, z(s))\, ds, \quad z_n(t) = T_n(t) z_0 + \int_0^t T_n(t-s) F(q_n, s, z_n(s))\, ds.$$

Consequently,

$$\begin{aligned}
\|z(t) - z_n(t)\|_{q,\delta} &= \|(-A)^\delta z(t) - (-A)^\delta z_n(t)\|_q \\
&\leq \left\|(-A)^\delta \left(T(t) - T_n(t)\right) z_0\right\|_q \\
&\quad + \left\|\int_0^t (-A)^\delta T(t-s) F(q, s, z(s)) - (-A)^\delta T_n(t-s) F(q_n, s, z_n(s))\, ds\right\|_q \\
&\leq \left\|(-A)^\delta \left(T(t) - T_n(t)\right) z_0\right\|_q \\
&\quad + \left\|\int_o^t (-A)^\delta T(t-s) F(q, s, z(s)) - (-A)^\delta T_n(t-s) F(q, s, z(s))\, ds\right\|_q \\
&\quad + \left\|\int_0^t (-A)^\delta T_n(t-s) \left[F(q, s, z(s)) - F(q_n, s, z(s))\right] ds\right\|_q \\
&\quad + \left\|\int_0^t (-A)^\delta T_n(t-s) \left[F(q_n, s, z(s)) - F(q_n, s, z_n(s))\right] ds\right\|_q \\
&\doteq I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t).
\end{aligned}$$

Note that, even when this last inequality holds only for t in $[0, \min\{t_1', t_1^n\})$, the real valued functions $I_1^n(t)$, $I_2^n(t)$ and $I_3^n(t)$ are well defined on $[0, t_1']$.

The following estimates hold:

$$\begin{aligned}
I_3^n(t) &\leq \int_0^t \|(-A)^\delta T_n(t-s)\|_{\mathcal{L}(Z_q)} \|F(q, s, z(s)) - F(q_n, s, z(s))\|_q\, ds \\
&\leq C_1 \int_0^t \|(-A_n)^\delta T_n(t-s)\|_{\mathcal{L}(Z_{q_n})} \|F(q, s, z(s)) - F(q_n, s, z(s))\|_q\, ds \\
&\leq C_1 \int_0^t \frac{C_\delta}{(t-s)^\delta} \|F(q, s, z(s)) - F(q_n, s, z(s))\|_q\, ds.
\end{aligned}$$

The second and third inequality follow from Lemma 3.3 and the Remark previous to Lemma 3.4, respectively. Now, $\|F(q, s, z(s)) - F(q_n, s, z(s))\|_q \to 0$ as $n \to \infty$ and there exists a constant C_2 independent of n such that

$\|F(q,s,z(s)) - F(q_n,s,z(s))\|_q \le C_2$ for every $s \in [0,t_1']$. These assertions follow easily from the continuity of $z(s)$ and the definition of the function F. Therefore, $I_3^n(t) \to 0$ as $n \to \infty$ on $[0,t_1']$ by the Dominated Convergence Theorem and also $I_3^n(t) \le \dfrac{C_1C_2C_\delta}{1-\delta} t^{1-\delta}$, $\forall n \in \mathbb{N}$, $\forall t \in [0,t_1']$.

To estimate $I_2^n(t)$, observe that

$$I_2^n(t) \le \int_0^t \|(-A)^\delta \left(T(t-s) - T_n(t-s)\right) F(q,s,z(s))\|_q \, ds.$$

Now, $\|F(q,s,z(s))\|_q$ is bounded on $[0,t_1']$, say $\|F(q,s,z(s))\|_q \le C_3$, $\forall t \in [0,t_1']$ and

$$\begin{aligned}
\|(-A)^\delta(T(t-s) - &T_n(t-s))\|_{\mathcal{L}(Z_q)} \\
&\le \|(-A)^\delta T(t-s)\|_{\mathcal{L}(Z_q)} + \|(-A)^\delta T_n(t-s)\|_{\mathcal{L}(Z_q)} \\
&\le \|(-A)^\delta T(t-s)\|_{\mathcal{L}(Z_q)} + C\|(-A_n)^\delta T_n(t-s)\|_{\mathcal{L}(Z_{q_n})} \\
&\le \frac{C_\delta}{(t-s)^\delta} + \frac{C\, C_\delta}{(t-s)^\delta} \doteq \frac{C_4}{(t-s)^\delta}.
\end{aligned}$$

On the other hand, for any $s \in [0,t_1']$ we have $\|(-A)^\delta \left(T(t-s) - T_n(t-s)\right) F(q,s,z(s))\|_q \to 0$ as $n \to \infty$ by Lemma 3.2. Therefore $I_2^n(t) \to 0$ as $n \to \infty$ by the Dominated Convergence Theorem and also $I_2^n(t) \le \dfrac{C_3C_4}{1-\delta} t^{1-\delta}$, $\forall n \in \mathbb{N}$, $\forall t \in [0,t_1']$.

In regard to $I_1^n(t)$ observe that by Lemma 3.2, $I_1^n(t) \to 0$ as $n \to \infty$ and also

$$\begin{aligned}
I_1^n(t) &= \left\|(-A)^\delta \left(T_n(t) - T(t)\right) z_0\right\|_q \\
&= \left\|(-A)^\delta (-A_n)^{-\delta} (-A_n)^\delta T_n(t) z_0 - (-A)^\delta T(t) z_0\right\|_q \\
&\le C \left\|T_n(t)(-A_n)^\delta z_0\right\|_q + \left\|T(t)(-A)^\delta z_0\right\|_q \\
&\le C \left\|T_n(t)\right\|_{\mathcal{L}(Z_q)} C \left\|(-A)^\delta z_0\right\|_q + \|T(t)\|_{\mathcal{L}(Z_q)} \left\|(-A)^\delta z_0\right\|_q \\
&\le C_5 \left\|(-A)^\delta z_0\right\|_q,
\end{aligned}$$

where we have used the fact that $z_0 \in D\left((-A)^\delta\right)$.

Similarly,

$$\begin{aligned}
I_4^n(t) &\le \int_0^t \|(-A)^\delta T_n(t-s)\|_{\mathcal{L}(Z_q)} \|F(q_n,s,z(s)) - F(q_n,s,z_n(s))\|_q \, ds \\
&\le C_6 \int_0^t \frac{1}{(t-s)^\delta} \|F(q_n,s,z(s)) - F(q_n,s,z_n(s))\|_q \, ds.
\end{aligned}$$

From the above estimates on $I_1^n(t)$, $I_2^n(t)$, $I_3^n(t)$ and $I_4^n(t)$,

$$\|z(t) - z_n(t)\|_{q,\delta} \le \epsilon_n(t) + C_6 \int_0^t \frac{1}{(t-s)^\delta} \|F(q_n,s,z(s)) - F(q_n,s,z_n(s))\|_q \, ds \tag{3.4}$$

for every $t \in [0, \min\{t_1', t_1^n\})$, where, for all $t \in [0,t_1']$, $\epsilon_n(t) \doteq I_1^n(t) + I_2^n(t) + I_3^n(t)$ satisfies $0 \le \epsilon_n(t) \le C_7$ for all $n \in \mathbb{N}$ and $\epsilon_n(t) \to 0$ as $n \to \infty$. In particular, $\int_0^{t_1'} \epsilon_n(t)\, dt \to 0$ as $n \to \infty$.

Let $K = K(\delta)$ be as in Lemma 3.4 and let us define $\tilde{K} \doteq C_7 + C_6 C_7 K$ and $M \doteq \sup_{0 \le t \le t_1'} \|z(t)\|_{q,\delta}$. From the continuity of $z(t)$ it follows that $M < \infty$ since $t_1' < t_1$. Since $z(0) = z_n(0) = z_0$, for each $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that $\|z_n(t)\|_{q,\delta} \le M + 2\tilde{K}$ for all $t \in [0,\delta_n]$. Let L be the Lipschitz constant of Theorem 2.4(i) for F, corresponding to the set $U \doteq [0,t_1'] \times \left\{\|z\|_{q,\delta} \le M + 2\tilde{K}\right\}$, for q and all the q_n's. Then, from (3.4) and Lemma 3.4, we have

$$\|z_n(t) - z(t)\|_{q,\delta} \le f_n(t) \quad \text{on } 0 \le t \le \delta_n, \tag{3.5}$$

where $f_n(t) \doteq \epsilon_n(t) + C_6 L K \displaystyle\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\, ds$, for $t \in [0,t_1']$.

Now,

$$\begin{aligned}\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\,ds &\le \int_0^t \frac{C_7}{(t-s)^\delta}\,ds\\ &= C_7\int_0^t \frac{1}{s^\delta}\,ds\\ &= \frac{C_7}{1-\delta}t^{1-\delta}.\end{aligned}$$

Choosing $\eta = \eta(L) > 0$ sufficiently small so that $t^{1-\delta} \le \dfrac{1-\delta}{2L}$ for every $t \in [0,\eta]$, it follows that

$$\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\,ds \le \frac{C_7}{2L} \quad \text{for every } t \in [0,\eta]. \tag{3.6}$$

On the other hand, if $\eta < t \le t_1'$

$$\begin{aligned}\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\,ds &= \int_0^t \frac{\epsilon_n(t-s)}{s^\delta}\,ds\\ &= \int_0^\eta \frac{\epsilon_n(t-s)}{s^\delta}\,ds + \int_\eta^t \frac{\epsilon_n(t-s)}{s^\delta}\,ds\\ &\le \frac{C_7}{1-\delta}\eta^{1-\delta} + \int_\eta^t \frac{\epsilon_n(t-s)}{\eta^\delta}\,ds\\ &\le \frac{C_7}{2L} + \frac{1}{\eta^\delta}\int_0^t \epsilon_n(t-s)\,ds\\ &\le \frac{C_7}{2L} + \frac{1}{\eta^\delta}\int_0^{t_1'} \epsilon_n(s)\,ds.\end{aligned}$$

Hence, since $\int_0^{t_1'} \epsilon_n(s)\,ds \to 0$, there exists N_0 such that

$$\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\,ds \le \frac{C_7}{2L} + \frac{C_7}{2L} = \frac{C_7}{L} \quad \forall t \in [\eta, t_1'] \text{ and } n \ge N_0. \tag{3.7}$$

By virtue of (3.6) and (3.7) one has

$$f_n(t) \le C_7 + C_6C_7K = \tilde{K} \qquad \forall t \in [0, t_1'] \text{ and } n \ge N_0. \tag{3.8}$$

Consequently, from (3.5) and (3.8),

$$\|z_n(t) - z(t)\|_{q,\delta} \le \tilde{K} \quad \forall n \ge N_0 \text{ and } t \in [0,\delta_n],$$

which implies

$$\|z_n(t)\|_{q,\delta} \le M + \tilde{K} \quad \forall n \ge N_0 \text{ and } t \in [0,\delta_n].$$

Finally, let $n \ge N_0$ be fixed. Then $z_n(t)$ exists on $[0,t_1']$ and for $t \in [0,t_1']$, $\|z_n(t)\|_{q,\delta} \le M + 2\tilde{K}$. In fact, suppose, on the contrary, that there exists $t^* < t_1'$ such that $\|z_n(t^*)\|_{q,\delta} = M + 2\tilde{K}$ and $\|z_n(t)\|_{q,\delta} < M + 2\tilde{K}$ for $0 < t < t^*$. Then, from (3.5) $\|z_n(t) - z(t)\|_{q,\delta} \le f_n(t) \le \tilde{K}$ on $[0,t^*)$ and therefore $\|z_n(t)\|_{q,\delta} \le M + \tilde{K}$ on $[0,t^*)$. By the continuity of $z_n(t)$, we must have $\|z_n(t^*)\|_{q,\delta} \le M + \tilde{K}$, which contradicts $\|z_n(t^*)\|_{q,\delta} = M + 2\tilde{K}$. The theorem then follows by taking $D = M + 2\tilde{K}$. ■

Theorem 3.6. *(Parameter Continuity) Under the same hypotheses of Theorem 3.5*

$$\|z_n(t) - z(t)\|_{q,\delta} \to 0, \quad \textit{as } n \to \infty$$

for every $t \in [0, t_1)$.

Proof. Let $\delta \in \left(\frac{3}{4}, 1\right)$ and $t_1' < t_1$. By Theorem 3.5 there exist constants $N_0 \in \mathbb{N}$ and $D > 0$ such that for every $n \geq N_0$, $z_n(t)$ exists on $[0, t_1']$ and satisfies $\|z_n(t)\|_{q,\delta} \leq D$ on that interval. Following the same steps of Theorem 3.5 we find that for every $t \in [0, t_1']$ and $n \geq N_0$

$$\begin{aligned}\|z(t) - z_n(t)\|_{q,\delta} &\leq \epsilon_n(t) + C_6 \int_0^t \frac{1}{(t-s)^\delta} \|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q \, ds \\ &\leq \epsilon_n(t) + LC_6 \int_0^t \frac{1}{(t-s)^\delta} \|z(s) - z_n(s)\|_{q,\delta} \, ds\end{aligned}$$

where $0 \leq \epsilon_n(t) \leq C_7$ and $\epsilon_n(t) \to 0$ as $n \to \infty$ for every $t \in [0, t_1']$. In the last inequality we have used the fact that F is locally Lipschitz continuous and $\|z_n(t)\|_{q,\delta} \leq D$, $\forall n \geq N_0$, $\forall t \in [0, t_1']$, as it was shown in Theorem 3.5.

Hence, by Lemma 3.4, there exists $K > 0$ such that

$$\|z(t) - z_n(t)\|_{q,\delta} \leq \epsilon_n(t) + KLC_6 \int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta} \, ds \longrightarrow 0 \quad \text{as } n \to \infty \qquad \forall t \in [0, t_1'].$$

Since t_1' is arbitrary, the theorem follows. ■

4. Parameter Identifiability

As we mentioned in the introduction, the parameter identifiability is an important issue when solving ID problems. Roughly speaking, the parameter identifiability can be thought of as the continuity and local invertibility of the mapping $q \longrightarrow z(\cdot\,; q)$ from the space $\mathcal{Q}$ of admissible parameters into the space of solutions (see [1]). Following are two results on this matter.

Theorem 4.1. *Let* $\delta \in \left(\frac{3}{4}, 1\right)$, $q = (\rho, C_v, \beta, \gamma, \alpha_2, \alpha_4, \alpha_6, \theta_1) \in \mathcal{Q}$, *and* $\tilde{q} = (\tilde{\rho}, \tilde{C}_v, \tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}_2, \tilde{\alpha}_4, \tilde{\alpha}_6, \tilde{\theta}_1) \in \mathcal{Q}$ *with* $\rho = \tilde{\rho}$, $C_v = \tilde{C}_v$, $\beta = \tilde{\beta}$, *and* $\gamma = \tilde{\gamma}$. *Let* $z_0(x) = \begin{pmatrix} u_0(x) \\ v_0(x) \\ w_0(x) \end{pmatrix} \in D\left((-A(q))^\delta\right)$ *and suppose that the solutions* $z(t; q)$ *and* $z(t; \tilde{q})$ *of (2.1) coincide on* $T_1 \leq t \leq T_2$ *for some* $0 \leq T_1 < T_2$. *Assume further that* $z(t; q) = \begin{pmatrix} u(\cdot\,, t) \\ v(\cdot\,, t) \\ w(\cdot\,, t) \end{pmatrix}$ *is such that* $v(\cdot\,, t^*) \not\equiv 0$ *for some* $t^* \in (T_1, T_2)$. *Then* $q = \tilde{q}$.

Remark. Note that the hypotheses of this theorem are satisfied if $T_1 = 0$ and the initial condition $z_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}$ is chosen such that $v_0 \not\equiv 0$.

Proof. Since $v(\cdot\,, t^*) \not\equiv 0$ and $v(0, t^*) = v(1, t^*) = 0$ there exists $\hat{x} \in \Omega$ such that $v_x(\hat{x}, t^*) \neq 0$. By continuity $v_x(\hat{x}, t) \neq 0$ on $t^* - \epsilon < t < t^* + \epsilon$ for some $\epsilon > 0$. Given that $v_x = u_{xt}$ there must exist $\hat{t} \in (t^* - \epsilon, t^* + \epsilon)$ such that

$$u_x(\hat{x}, \hat{t}) \neq 0. \tag{4.1}$$

On the other hand,

$$\dot{z}(t; q) = A(q) z(t; q) + F(q, t, z(t; q)) \text{ and } \dot{z}(t; \tilde{q}) = A(\tilde{q}) z(t; \tilde{q}) + F(\tilde{q}, t, z(t; \tilde{q})) \quad \forall t \in [T_1, T_2]. \tag{4.2}$$

But, $\rho = \tilde{\rho}$, $C_v = \tilde{C}_v$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ imply $A(q) = A(\tilde{q})$ since the operator $A(q)$ does not depend on any of the rest of the variables in q. Thus, since $z(t; q) = z(t; \tilde{q})$ $\forall t \in [T_1, T_2]$ equation (4.2) imply

$$F(q, t, z(t; q)) = F(\tilde{q}, t, z(t; \tilde{q})) \quad \forall t \in [T_1, T_2],$$

and, using the definition of $F(q,t,z)$,

$$\frac{\partial}{\partial x}\left[2\alpha_2(w(x,t)\ +\ L(x,t)-\theta_1)u_x(x,t)-4\alpha_4 u_x(x,t)^3+6\alpha_6 u_x(x,t)^5\right]$$
$$=\frac{\partial}{\partial x}\left[2\tilde{\alpha}_2(w(x,t)+L(x,t)-\tilde{\theta}_1)u_x(x,t)-4\tilde{\alpha}_4 u_x(x,t)^3+6\tilde{\alpha}_6 u_x(x,t)^5\right] \tag{4.3a}$$

and

$$\alpha_2(w(x,t)+L(x,t))u_x(x,t)v_x(x,t)=\tilde{\alpha}_2(w(x,t)+L(x,t))u_x(x,t)v_x(x,t) \tag{4.3b}$$

for all $x\in\Omega$, $t\in[T_1,T_2]$.

From (4.3b) it follows that

$$(\alpha_2-\tilde{\alpha}_2)\theta(x,t)u_x(x,t)v_x(x,t)=0,\quad \forall x\in\Omega,\ t\in[T_1,T_2]$$

where $\theta(x,t)=w(x,t)+L(x,t)=w(x,t)+\theta_\Gamma(t)\cos(2\pi x)$. Since $u_x(\hat{x},\hat{t})v_x(\hat{x},\hat{t})\neq 0$ and $\theta>0$, we conclude that $\alpha_2=\tilde{\alpha}_2$. Consequently, equation (4.3a) now reads

$$u_{xx}(x,t)\left(2\alpha_2(\theta_1-\tilde{\theta}_1)+12(\alpha_4-\tilde{\alpha}_4)u_x(x,t)^2-30(\alpha_6-\tilde{\alpha}_6)u_x(x,t)^4\right)=0 \tag{4.4}$$

$\forall x\in\Omega,\ t\in[T_1,T_2]$.

Now, if $u_{xx}(x,\hat{t})$ were identically equal to zero on Ω then, using the boundary conditions, we would have $u(\cdot,\hat{t})\equiv 0$ and $u_x(\cdot,\hat{t})\equiv 0$ which obviously contradicts (4.1). Therefore, there must exist a,b, $0<a<b<1$ such that $u_{xx}(x,\hat{t})\neq 0$ for $a<x<b$, which implies $u_x(x,\hat{t})$ cannot be constant on (a,b). Therefore the functions 1, $u_x(x,\hat{t})^2$ and $u_x(x,\hat{t})^4$ are linearly independent as functions of x on (a,b). Hence, from (4.4) we obtain $\alpha_4=\tilde{\alpha}_4$, $\alpha_6=\tilde{\alpha}_6$ and $\theta_1=\tilde{\theta}_1$, and the theorem follows.

■

Under slightly more restrictive hypotheses as those of Theorem 4.1, it is possible to obtain one-to-oneness also with respect to the parameter γ. The following theorem shows this result.

Theorem 4.2. *Let $\delta\in(\frac{3}{4},1)$, $q,\tilde{q}\in\mathcal{Q}$ with $\rho=\tilde{\rho}$, $C_v=\tilde{C}_v$, $\beta=\tilde{\beta}$; $z_0(x)=\begin{pmatrix}u_0(x)\\ v_0(x)\\ w_0(x)\end{pmatrix}\in D\left((-A(q)^\delta)\right)$ and suppose that the solutions $z(t;q)$ and $z(t;\tilde{q})$ of (2.1) coincide on $T_1\le t\le T_2$ for some $0\le T_1<T_2$. Assume further that $v(\cdot,\hat{t})\not\equiv 0$ for some $\hat{t}\in[T_1,T_2]$. If, in addition either of the following two conditions hold, then $q=\tilde{q}$.*

- **(i)** *There exists $t^*\in[T_1,T_2]$ such that $u_{xxxx}(0,t^*)\neq 0$ or $u_{xxxx}(1,t^*)\neq 0$.*
- **(ii)** *There exists $t^*\in[T_1,T_2]$ such that the functions $u_{xxxx}(\cdot,t^*)$, $u_{xx}(\cdot,t^*)$, $u_{xx}(\cdot,t^*)u_x(\cdot,t^*)^2$, $u_{xx}(\cdot,t^*)u_x(\cdot,t^*)^4$ are linearly independent as functions of x on Ω.*

Proof. Following the same steps as in Theorem 4.1, the functions $u(x,t)$, $v(x,t)$, $\theta(x,t)$ must satisfy

$$\frac{\gamma}{\rho}u_{xxxx}(x,t)+\frac{\partial}{\partial x}\left[2\alpha_2(\theta(x,t)-\theta_1)u_x(x,t)-4\alpha_4 u_x(x,t)^3+6\alpha_6 u_x(x,t)^5\right]$$
$$=\frac{\tilde{\gamma}}{\rho}u_{xxxx}(x,t)+\frac{\partial}{\partial x}\left[2\tilde{\alpha}_2(\theta(x,t)-\tilde{\theta}_1)u_x(x,t)-4\tilde{\alpha}_4 u_x(x,t)^3+6\tilde{\alpha}_6 u_x(x,t)^5\right] \tag{4.5a}$$

and

$$(\alpha_2-\tilde{\alpha}_2)\theta(x,t)u_x(x,t)v_x(x,t)=0,\quad \forall x\in\Omega \text{ and } t\in[T_1,T_2]\,. \tag{4.5b}$$

As in Theorem 4.1, equation (4.5b) implies $\alpha_2=\tilde{\alpha}_2$ and consequently (4.5a) yields

$$\frac{(\gamma-\tilde{\gamma})}{\rho}u_{xxxx}(x,t)$$
$$+\,u_{xx}(x,t)\left(2\alpha_2(\theta_1-\tilde{\theta}_1)+12(\alpha_4-\tilde{\alpha}_4)u_x(x,t)^2-30(\alpha_6-\tilde{\alpha}_6)u_x(x,t)^4\right)=0 \tag{4.6}$$

$\forall x\in\Omega,\ t\in[T_1,T_2]$.

Suppose condition (i) holds and wlog assume $u_{xxxx}(0,t^*) \neq 0$. Then, evaluating equation (4.6) at $x=0$ and $t=t^*$ yields $\frac{(\gamma-\tilde{\gamma})}{\rho}u_{xxxx}(0,t^*)=0$, which implies $\gamma=\tilde{\gamma}$. Equation (4.6) now takes the form

$$u_{xx}(x,t)\left(2\alpha_2(\theta_1-\tilde{\theta}_1)+12(\alpha_4-\tilde{\alpha}_4)u_x(x,t)^2-30(\alpha_6-\tilde{\alpha}_6)u_x(x,t)^4\right)=0$$

$\forall x \in \Omega$, $\forall t \in [T_1,T_2]$. Following the same reasoning as in Theorem 4.1, the above identity implies $\theta_1=\tilde{\theta}_1$, $\alpha_4=\tilde{\alpha}_4$, $\alpha_6=\tilde{\alpha}_6$.

On the other hand, if condition (ii) holds then the result follows immediately from equation (4.6) and the linear independence of the functions $u_{xxxx}(\cdot,t^*)$, $u_{xx}(\cdot,t^*)$, $u_{xx}(\cdot,t^*)u_x(\cdot,t^*)^2$ and $u_{xx}(\cdot,t^*)u_x(\cdot,t^*)^4$. ■

Theorems 4.1 and 4.2 together with the continuity results of section 3 imply, under the appropriate hypotheses, the identifiability of problem (1.1) with respect to the parameters that define the free energy of the system.

5. Conclusions

In this paper we have shown that the solutions of the IBVP (1.1), with free energy potential Ψ in the Landau-Ginzburg form (1.2), depend continuously on the parameters ρ, C_v, β,α_2, α_4, α_6, θ_1 and γ. In particular, we have shown that if $\{q_n = (\rho_n, C_{v,n}, \beta_n, \alpha_{2,n}, \alpha_{4,n}, \alpha_{6,n}, \theta_{1,n}, \gamma_n)\}_{n=1}^{\infty}$ is a sequence of admissible parameters converging to the admissible parameter q, then not only $z(t;q_n) \to z(t;q)$ in the norm of Z_q, but also in the stronger $\|\cdot\|_{q,\delta}$-norm ($\delta=\frac{3}{4}+\epsilon$). We have also shown that under rather weak hypotheses, the free energy potential Ψ as given by (1.2) is identifiable from the IBVP (1.1). More precisely, if the conditions of Theorem 4.2 hold, then the mapping $(\alpha_2,\alpha_4,\alpha_6,\theta_1,\gamma) \longrightarrow z(\cdot\,;q)$ is continuous and invertible. Although this is a partial result since it does not imply the invertibility of the mapping $q \longrightarrow z(\cdot\,;q)$, it is appropriate to emphasize its importance from a practical point of view in the sense that the parameters α_2, α_4, α_6, θ_1 and γ are the only non-physical parameters appearing in the model.

References

[1] BANKS, H. T., and KUNISCH, K., *"Estimation Techniques for Distributed Parameters Systems"*, Birkhäuser, 1989.

[2] BENSOUSSAN A., DA PRATO G., DELFOUR M. and MITTER S., *"Representation and Control of Infinite Dimensional Sytems"*, Vol. I Birkhäuser, 1992.

[3] BURNS, J. A. and SPIES, R. D., *"Finite Element Approximation of a Shape Memory Alloy"*, Proceedings of the ADPA/AIAA/ASME/SPIE Conference on Active Materials and Adaptive Structures, Alexandria, Virginia, 1991.

[4] BURNS, J. A. and SPIES, R. D., *"Modelling for Control of Shape Memory Alloys"*, 30^{th} IEEE Conference on Decision and Control, Brighton, England, 1991.

[5] BURNS, J. A. and SPIES, R. D., *" A Numerical Study of Parameter Sensitivities in Landau-Ginzburg Models of Phase Transitions in Shape Memory Alloys"*, Journal of Intelligent Material Systems and Structures, vol. 5, May 1994, pp 321-332.

[6] CHEN, G. and RUSSELL, D. L., *"A Mathematical Model for Linear Elastic Systems with Structural Damping"*, Quarterly of Applied Mathematics, January 1982, pp 433-454.

[7] CHEN, S. and TRIGGIANI, R., *"Proof of Extensions of Two Conjectures on Structural Damping for Elastic Systems"*, Pacific Journal of Mathematics, Vol. 136, No 1, 1989, pp 15-55.

[8] DAFERMOS, C. M., *"Global Smooth Solutions to the Initial-Boundary Value Problem for the Equations of One-Dimensional Nonlinear Thermoviscoelasticity"*, Siam J. Math. Anal., Vol. 13, No 3, pp 397-408, May 1982.

[9] DAFERMOS, C. M. and HSIAO, L., *"Global Smooth Thermomechanical Processes in One-Dimensional Thermoviscoelasticity"*, Nonlinear Anal. Theory Methods Appl., Vol. 6, 1982, pp 435-454.

[10] DELAEY, L. and CHANDRASEKARAN, M., *"Proceeding, International Conference on Martensitic Transformations"*, Les Editions de Physique, Les Ulis, 1984.

[11] DELAEY, L., KRISHNAN, R. V., TAS, H. and WARLIMONT, H., *"Thermoelasticity, Pseudoelasticity and the Memory Effects Associated with Martensitic Transformations"*, Journal of Materials Science, Vol. 9, 1974, pp 1521-1555.

[12] FALK, F., *"One Dimensional Model of Shape Memory Alloys"*, Arch. Mech., 35, pp 63-84, Warszawa, (1983).

[13] FUNAKUBO, H. (Ed.), *"Shape Memory Alloys"*, Precision Machinery and Robotics, Vol. 1, Translated from the Japanese by J. B. Kennedy, Gordon and Breach Science Publishers, (1987).

[14] GURTIN, M. E., *"On the Thermodynamics of Materials with Memory"*, Arch Rational Mech. Anal., Vol. 28, 1968, pp 40-50.

[15] HENRY, D. *"Geometric Theory of Semilinear Parabolic Equations"*, Springer-Verlag, 1989.

[16] HOFFMANN, K. H. and SONGMU, Z., *"Uniqueness for Nonlinear Coupled Equations Arising from Alloy Mechanism"*, Preprint 118, Institut für Mathematik, Ausburg, (1986).

[17] HOFFMANN, K. H. and SONGMU, Z., *"Uniqueness for Structural Phase Transitions in Shape Memory Alloys"*, Mathematical Methods in the Applied sciences, Vol. 10, pp 145-151, (1988).

[18] KATO, T., *"Perturbation Theory for Linear Operators"*, Second Edition, Springer-Verlag, 1980.

[19] LIONS, J. L. and MAGENES E., *"Non-Homogeneous Boundary Value Problems and Applications"*, Vol. I, Springer-Verlag, 1972

[20] MORIN, P. and SPIES, R. D., *"Convergent Spectral Approximations for the Thermomechanical Processes in Shape Memory Alloys"*, in preparation.

[21] MULLER, I. and WILMANSKI, K., *"A model for Phase Transitions in Pseudoelastic Bodies"*, Il Nuovo Cimento, Vol. 57B, No 2, pp 283-318, (1980).

[22] NIEZGODKA, M. and SPREKELS, J., *"Existence of Solutions for a Mathematical Model of Structural Phase Transitions in Shape Memory Alloys"*, Mathematical Methods in the Applied Sciences, Vol. 10, pp 197-223, (1988).

[23] NIEZGODKA, M., SONGMU, Z. and SPREKELS, J., *"Global Solutions to a Model of Structural Phase Transitions in Shape Memory Alloys"*, Journal of Mathematical Analysis and Applications, Vol. 130, 1988, pp 39-54.

[24] PAZY, A., *"Semigroups of Linear Operators and Applications to Partial Differential Equations"*, Corrected Second Printing, Springer-Verlag, 1983.

[25] RUSSELL, D. L., *"On the Positive Square Root of the Fourth Derivative Operator"*, Quarterly of Applied Mathematics, Vol. XLVI, No 4, December 1988, pp 751-773.

[26] SONGMU, Z., *"Global Solutions to the Thermomechanical Equations with Non-convex Landau-Ginzburg Free Energy"*, Journal of Applied Mathematics and Physics (ZAMP), Vol. 40, pp 111-127, January 1989.

[27] SONGMU, Z. and SPREKELS, J., *"Global Solutions to the Equations of a Ginzburg-Landau Theory for Structural Phase Transitions in Shape Memory Alloys"*, Physica D, Vol. 39, pp 59-76, (1989).

[28] SPIES, R. D., *"A State-Space Approach to a One-Dimensional Mathematical Model for the Dynamics of Phase Transitions in Pseudoelastic Materials"*, Journal of Mathematical Analysis and Applications, 190, (1995), pp 58-100.

[29] SPIES, R. D., *"Results on a Mathematical Model of Thermomechanical Phase Transitions in Shape Memory Materials"*, Journal of Smart Materials and Structures, 3, (1994), pp 459-469.

[30] SPREKELS, J., *"Global Existence for Thermomechanical Processes with Nonconvex Free Energies of Ginzburg-Landau Form"*, Journal of Mathematical Analysis and Applications, Vol. 141, pp 333-348, (1989).

[31] TRIEBEL, H., *"Interpolation Theory, Function Spaces, Differential Operators"*, North-Holland Publishing Company, 1978.

[32] TRIGGIANI, R ., *"On the Stabilizability Problem in Banach Space"*, J. Math. Anal. Appl. 52, pp 383-403, (1975).

[33] WLOKA, J., *"Partial Differential Equations"*, Cambridge University Press, 1987.