

# Parameter Differentiability of the Solution of a Nonlinear Abstract Cauchy Problem

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**Abstract:** In this article, an abstract nonlinear evolution equation with a parameter appearing in the nonlinear part is considered. Using the theory of analytic semigroups and a generalization of Gronwall's lemma for singular kernels, sufficient conditions to ensure differentiability of the solution with respect to the parameter are derived in terms of the smoothness of the nonlinear part of the equation. The results are applied to a nonlinear system of PDE's modeling the dynamics of shape memory alloys.

## 1. Introduction

Let  $Z$  and  $\tilde{Q}$  be two Banach spaces,  $A$  the infinitesimal generator of an analytic semigroup  $T(t)$  on  $Z$ ,  $D$  a subset of  $Z$ ,  $\mathcal{Q}$  an open subset of  $\tilde{Q}$ ,  $T > 0$  and  $F : \mathcal{Q} \times [0, T] \times D \rightarrow Z$ . We shall consider the following nonlinear Cauchy problem in  $Z$ :

$$\begin{cases} \dot{z}(t) = Az(t) + F(q, t, z(t)), & t \in (0, T) \\ z(0) = z_0. \end{cases} \quad (1.1)$$

The spaces  $Z$  and  $\tilde{Q}$  will be referred to as the state-space and the parameter space, respectively, while  $\mathcal{Q}$  will be called the admissible parameter set.

For the application of quasilinearization algorithms in parameter identification for systems like (1.1) and other similar types of equations ([5], [7], [9]) it is essential to obtain conditions under which the solution is differentiable with respect to the parameter  $q$ .

In [6] this differentiability problem was analyzed in the context of linear abstract Cauchy problems of the type

$$\dot{z}(t) = A(q)z(t) + u(t), \quad z(0) = z_0 \quad (1.2)$$

in which  $A(q)$  generates a strongly continuous semigroup and can be written in the form  $A(q) = A + B(q)$  where  $B(q)$  is bounded. Later on ([4]), the same problem was studied under weaker assumptions. Here, the parameter  $q$  was not restricted to appear in a bounded term of the operator  $A(q)$ .

In the present article we shall prove the  $q$ -differentiability of the solution of system (1.1). Several problems in diverse areas like in the study of flexible and smart structures, Geology, Ecology, Chemical Engineering,

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<sup>◊</sup>This research was supported in part by the Air Force Office of Scientific Research under grants F49620-93-1-0280 and F49620-96-1-0329.

<sup>†</sup>The work of the authors was supported in part by CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas of Argentina and UNL, Universidad Nacional del Litoral through project CAI+D 94-0016-004-023.

Biology, etc. (see [3], [9]) originate problems whose abstract formulation result in nonlinear systems in which the unknown vector parameter appears in the nonlinear term. In these and many other practical problems it is very important to have a mathematical framework in which the identification of the parameter be possible from experimental and laboratory data.

The quasilinearization methods for identification ([5],[7]) provide practical mathematical tools for achieving this goal. However, their application require not only that the solution depends continuously on the unknown parameter but also that an explicit formula for the derivative of the solution with respect to that parameter be derived.

The organization of the article is as follows. In Section 2 the  $q$ -differentiability of the solution of (1.1) is studied and sufficient conditions are given under which that property holds. In Section 3 a regularity result for that derivatives is obtained. The latter result is required in order to prove *convergence* of the quasilinearization algorithms previously mentioned while the former is required for the algorithm to be well-defined. In Section 4 an application example is considered in the context of a mathematical model for shape memory alloys. The partial differential equations that model the dynamics of these materials result in an abstract Cauchy problem in which the unknown parameters defining the free energy potential arise in the nonlinear terms. Theorems of Sections 2 and 3 are used in this particular case to show that the solution depends smoothly on the unknown vector parameter.

## 2. Parameter differentiability of the solution of (1.1)

In this section we will prove the Fréchet differentiability of the solution of (1.1) with respect to the parameter  $q$ . We first recall some properties of analytic semigroups and make some general assumptions on the nonlinear part of the equation.

Let  $\sigma(A)$  denote the spectrum of the operator  $A$ . Since  $A$  generates an analytic semigroup, the type of  $A$ , defined as  $\omega \doteq \sup \{ \operatorname{Re}(\lambda) : \lambda \in \sigma(A) \}$  is finite and for any  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > \omega$ , the fractional powers  $(\lambda I - A)^\delta$  of  $\lambda I - A$  are closed, linear and invertible operators in  $Z$  for any  $\delta \in [0, 1]$  (see [10]). In what follows,  $\lambda$  will be fixed,  $\operatorname{Re}(\lambda) > \omega$  and  $Z_\delta$  shall denote the space  $D((\lambda I - A)^\delta)$  imbedded with the norm of the graph of  $(\lambda I - A)^\delta$ . Due to the fact that  $\operatorname{Re}(\lambda) > \omega$ , one has that  $\lambda \in \rho(A)$ , the resolvent set of  $A$ , and the graph norm is equivalent to the norm  $\|z\|_\delta \doteq \|(\lambda I - A)^\delta z\|_Z$ .

Consider the following standing hypothesis.

**(H1).** *There exists  $\delta \in (0, 1)$  such that  $Z_\delta \subset D$  and  $F : \mathcal{Q} \times [0, T] \times Z_\delta \rightarrow Z$  is locally Lipschitz continuous in  $t$  and  $z$ , i.e., for any  $q \in \mathcal{Q}$  and any bounded subset  $U$  of  $[0, T] \times Z_\delta$  there exists a constant  $L = L(q, U)$  such that*

$$\|F(q, t_1, z_1) - F(q, t_2, z_2)\|_Z \leq L(|t_1 - t_2| + \|z_1 - z_2\|_\delta), \quad \forall (t_i, z_i) \in U.$$

where the constant  $L$  can be chosen independent of  $q$  on any compact subset  $\mathcal{Q}_C$  of  $\mathcal{Q}$ .

The following result follows immediately from Theorem 6.3.1 in [10].

**Theorem 2.1.** *Let  $q \in \mathcal{Q}$  and  $z_0 \in Z_\delta$  and assume  $F$  satisfies hypothesis (H1). Then there exists  $t_1 = t_1(q, z_0) > 0$  such that  $(P)_q$  has a unique classical solution on  $[0, t_1)$ , i.e., there exists a function  $z(\cdot) \in C^0([0, t_1) : Z_\delta) \cap C^1((0, t_1) : Z)$  such that*

$$\begin{cases} \dot{z}(t) = Az(t) + F(q, t, z(t)), & t \in (0, t_1) \\ z(0) = z_0. \end{cases}$$

The function  $z(t)$  satisfies the integral equation

$$z(t) = T(t)z_0 + \int_0^t T(t-s)F(q, s, z(s)) ds, \quad \forall t \in [0, t_1).$$

Also,  $t_1$  can be chosen positive independent of  $q$  on any compact subset  $\mathcal{Q}_C$  of  $\mathcal{Q}$ .

Let us denote by  $z(t; q)$  the solution  $z(t)$  of (1.1).

The following generalization of Gronwall's Lemma for singular kernels, whose proof can be found in [8] (Lemma 7.1.1), will be essential for the main result of this section.

**Lemma 2.2.** Suppose  $L \geq 0$ ,  $0 < \delta < 1$  and  $a(t)$  is a nonnegative, locally integrable function on  $0 \leq t \leq T$ . Let  $u(t)$  be a real valued function defined on  $[0, T]$  satisfying

$$u(t) \leq a(t) + L \int_0^t \frac{1}{(t-s)^\delta} u(s) ds$$

on this interval. Then, there exists a constant  $K = K(\delta)$  such that

$$u(t) \leq a(t) + KL \int_0^t \frac{a(s)}{(t-s)^\delta} ds \quad \text{for } 0 \leq t < T.$$

The following theorem states a result relating the regularity of  $F$  and the Lipschitz continuity of  $z(t; q)$  with respect to  $q$ .

**Theorem 2.3.** Suppose  $F(q, t, z)$  satisfies hypothesis (H1) for some  $\delta \in (0, 1)$ . If the mapping  $q \rightarrow F(q, \cdot, z)$  from  $\mathcal{Q}$  into  $L^\infty(0, T : Z)$  is locally Lipschitz continuous for all  $z \in Z_\delta$  with Lipschitz constant independent of  $z$  on  $Z_\delta$ -bounded sets, then the mapping  $q \rightarrow z(\cdot; q)$  is locally Lipschitz continuous from  $\mathcal{Q}$  into  $L^\infty(0, T : Z_\delta)$ .

*Proof.* Let  $t \in [0, T]$  and  $q_1, q_2 \in \mathcal{Q}$ . Then

$$z(t; q_1) = T(t)z_0 + \int_0^t T(t-s)F(q_1, s, z(s; q_1)) ds,$$

$$z(t; q_2) = T(t)z_0 + \int_0^t T(t-s)F(q_2, s, z(s; q_2)) ds$$

and therefore

$$\begin{aligned} z(t; q_1) - z(t; q_2) &= \int_0^t T(t-s) [F(q_1, s, z(s; q_1)) - F(q_2, s, z(s; q_2))] ds \\ &= \int_0^t T(t-s) [F(q_1, s, z(s; q_1)) - F(q_2, s, z(s; q_1))] ds \\ &\quad + \int_0^t T(t-s) [F(q_2, s, z(s; q_1)) - F(q_2, s, z(s; q_2))] ds. \end{aligned}$$

Hence, if  $\delta \in (0, 1)$  is such that (H1) holds, then it follows that

$$\begin{aligned} \|z(t; q_1) - z(t; q_2)\|_\delta &\leq \int_0^t \frac{C}{(t-s)^\delta} \|F(q_1, s, z(s; q_1)) - F(q_2, s, z(s; q_1))\| \\ &\quad + \int_0^t \frac{C}{(t-s)^\delta} \|F(q_2, s, z(s; q_1)) - F(q_2, s, z(s; q_2))\| \\ &\leq C_1 \|q_1 - q_2\| \frac{t^{1-\delta}}{1-\delta} + \int_0^t \frac{CL}{(t-s)^\delta} \|z(s; q_1) - z(s; q_2)\|_\delta ds \\ &\leq C_2 \|q_1 - q_2\| + \int_0^t \frac{CL}{(t-s)^\delta} \|z(s; q_1) - z(s; q_2)\|_\delta ds. \end{aligned}$$

Applying Lemma 2.2 we obtain

$$\|z(t; q_1) - z(t; q_2)\|_\delta \leq K_1 \|q_1 - q_2\| e^{K_2 T}$$

for some constants  $K_1, K_2$ . The theorem then follows. ■

Next, sufficient conditions on  $F$  are given that ensure differentiability of the mapping  $q \rightarrow z(t; q)$ .

**Theorem 2.4.** Assume **(H1)** holds for some  $\delta \in (0, 1)$  and the mapping  $(q, z(\cdot)) \rightarrow F(q, \cdot, z(\cdot))$  from  $\mathcal{Q} \times L^\infty(0, T : Z_\delta)$  into  $L^\infty(0, T : Z)$  is Fréchet differentiable in both variables. Assume also that the mapping  $(q, z(\cdot)) \rightarrow F_q(q, \cdot, z(\cdot))$  from  $\mathcal{Q} \times L^\infty(0, T : Z_\delta)$  into  $L^\infty(0, T : \mathcal{L}(\tilde{\mathcal{Q}}, Z))$  is locally Lipschitz continuous with respect to  $q$  and  $z$ .

Then the mapping  $q \rightarrow z(\cdot; q)$  is Fréchet differentiable from  $\mathcal{Q} \rightarrow L^\infty(0, T : Z_\delta)$  and for any  $h \in \tilde{\mathcal{Q}}$ ,  $z_q(t; q)h$  is the solution  $v_h(t)$  of the linear IVP

$$\begin{cases} \dot{v}_h(t) = Av_h(t) + F_z(q, t, z(t; q))v_h(t) + F_q(q, t, z(t; q))h, & t \in (0, T) \\ v_h(0) = 0. \end{cases}$$

*Proof.* Let  $t \in [0, T]$  be fixed and  $\delta \in (0, 1)$  as in **(H1)**. Then for any  $h \in \tilde{\mathcal{Q}}$  such that  $q + h \in \mathcal{Q}$  we have

$$\begin{aligned} z(t; q + h) &= T(t)z_0 + \int_0^t T(t-s)F(q + h, s, z(s; q + h)) ds, \\ z(t; q) &= T(t)z_0 + \int_0^t T(t-s)F(q, s, z(s; q)) ds \end{aligned}$$

and

$$v_h(t) = \int_0^t T(t-s) [F_z(q, s, z(s; q))v_h(s) + F_q(q, s, z(s; q))h] ds.$$

Let  $\epsilon > 0$ . We will show that there exists  $\gamma > 0$  such that if  $h \in \tilde{\mathcal{Q}}$  and  $\|h\| < \gamma$  then  $\|z(t; q + h) - z(t; q) - v_h(t)\|_\delta < \epsilon \|h\|$ . For this purpose, observe that

$$\begin{aligned} & z(t; q + h) - z(t; q) - v_h(t) \\ &= \int_0^t T(t-s) [F(q + h, s, z(s; q + h)) - F(q, s, z(s; q))] ds \\ &\quad - \int_0^t T(t-s) [F_z(q, s, z(s; q))v_h(s) + F_q(q, s, z(s; q))h] ds \\ &= \int_0^t T(t-s) [F(q + h, s, z(s; q + h)) - F(q + h, s, z(s; q))] ds \\ &\quad + \int_0^t T(t-s) [F(q + h, s, z(s; q)) - F(q, s, z(s; q)) - F_q(q, s, z(s; q))h] ds \\ &\quad + \int_0^t T(t-s) [F(q, s, z(s; q + h)) - F(q, s, z(s; q)) - F_z(q, s, z(s; q))v_h(s)] ds \\ &\quad + \int_0^t T(t-s) [F(q, s, z(s; q)) - F(q, s, z(s; q + h))] ds \\ &\doteq I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where  $I_i$ ,  $i = 1, 2, 3, 4$  denotes the  $i$ th term in the order given above.

Since  $F$  is Fréchet differentiable with respect to  $z$ , there exists  $\gamma_1 > 0$  such that if  $\|z(q + h; \cdot) - z(q; \cdot)\|_{L^\infty(0, T; Z_\delta)} < \gamma_1$ , then

$$\begin{aligned} & \|I_3\|_\delta \\ &\leq \int_0^t \frac{C}{(t-s)^\delta} \|F(q, s, z(s; q + h)) - F(q, s, z(s; q)) \\ &\quad - F_z(q, s, z(s; q))(z(s; q + h) - z(s; q))\|_Z ds \\ &\quad + \int_0^t \frac{C}{(t-s)^\delta} \|F_z(q, s, z(s; q))(z(s; q + h) - z(s; q) - v_h(s))\|_Z ds \\ &\leq \int_0^t \frac{C\epsilon}{(t-s)^\delta} \|z(s; q + h) - z(s; q)\|_\delta ds + \int_0^t \frac{C_1}{(t-s)^\delta} \|z(s; q + h) - z(s; q) - v_h(s)\|_\delta ds \end{aligned}$$

Now, by virtue of Theorem 2.3, there exist constants  $\gamma_2, K > 0$  such that  $\|z(\cdot; q+h) - z(\cdot; q)\|_{L^\infty(0,T;Z_\delta)} \leq K\|h\| < \gamma_1$  whenever  $\|h\| < \gamma_2$ . Hence, if  $\|h\| < \gamma_2$ , we obtain

$$\|I_3\|_\delta \leq \int_0^t \frac{C K \epsilon}{(t-s)^\delta} \|h\| ds + \int_0^t \frac{C_1}{(t-s)^\delta} \|z(s; q+h) - z(s; q) - v_h(s)\|_\delta ds.$$

Also, since  $F$  is Fréchet differentiable with respect to  $q$ , there exists  $\gamma_3 > 0$  such that if  $\|h\| < \gamma_3$ , then

$$\|I_2\|_\delta \leq \int_0^t \frac{C \epsilon}{(t-s)^\delta} \|h\| ds.$$

On the other hand, observe that  $I_1 + I_4$  can be written as

$$\begin{aligned} I_1 + I_4 &= \int_0^T T(t-s) [F(q+h, s, z(s; q+h)) - F(q, s, z(s; q+h))] ds \\ &\quad - \int_0^T T(t-s) [F(q+h, s, z(s; q)) - F(q, s, z(s; q))] ds \\ &= \int_0^t T(t-s) F_q(q + \alpha_1(h)h; s, z(s; q+h)) h ds \\ &\quad - \int_0^t T(t-s) F_q(q + \alpha_2(h)h; s, z(s; q)) h ds \end{aligned}$$

where  $0 < \alpha_1(h), \alpha_2(h) < 1$ . Consequently, by the Lipschitz continuity of  $F_q$  we have that

$$\begin{aligned} \|I_1 + I_4\|_\delta &\leq \int_0^t \frac{C}{(t-s)^\delta} \|F_q(q + \alpha_1(h)h, s, z(s; q+h)) - F_q(q + \alpha_1(h)h, s, z(s; q))\|_Z \|h\| ds \\ &\quad + \int_0^t \frac{C}{(t-s)^\delta} \|F_q(q + \alpha_1(h)h, s, z(s; q)) - F_q(q + \alpha_2(h)h, s, z(s; q))\|_Z \|h\| ds \\ &\leq \|h\| \int_0^t \frac{C}{(t-s)^\delta} (L_1 \|z(s; q+h) - z(s; q)\|_\delta + L_2 |\alpha_1(h) - \alpha_2(h)| \|h\|) ds \\ &\leq C_3 \|h\|^2, \end{aligned}$$

where the last inequality follows by virtue of Theorem 2.3.

Summarizing, there exist constants  $\gamma^*, K_1, K_2 > 0$  such that whenever  $\|h\| < \gamma^*$  one has

$$\|z(t; q+h) - z(t; q) - v_h(t)\|_\delta \leq K_1 \|h\| \epsilon + K_2 \int_0^t \frac{\|z(t; q+h) - z(t; q) - v_h(t)\|_\delta}{(t-s)^\delta} ds.$$

The above inequality together with Lemma 2.2 imply the existence of a constant  $\tilde{K}$  such that

$$\|z(t; q+h) - z(t; q) - v_h(t)\|_\delta \leq \tilde{K} \|h\| \epsilon, \quad \text{provided } \|h\| \leq \gamma^*,$$

and the theorem follows. ■

### 3. Lipschitz continuity of the Fréchet derivative

In order to prove convergence of the quasilinearization algorithms it is necessary not only that solutions be differentiable with respect to the unknown parameter  $q$  but also that the corresponding derivative be smooth. The next result provides sufficient conditions for the Lipschitz regularity of  $z_q$ .

**Theorem 3.1.** *Let the hypotheses of Theorem 2.4 hold. Assume also that the mapping  $(q, z(\cdot)) \rightarrow F_z(q, \cdot, z(\cdot))$  from  $\mathcal{Q} \times L^\infty(0, T : Z_\delta)$  into  $L^\infty(0, T : \mathcal{L}(Z_\delta, Z))$  is locally Lipschitz continuous with respect to both variables  $q$  and  $z(\cdot)$ . Then, the mapping  $q \rightarrow z_q(\cdot; q)$  from  $\mathcal{Q} \rightarrow L^\infty(0, T : \mathcal{L}(\tilde{\mathcal{Q}}, Z))$  is locally Lipschitz continuous.*

*Proof.* By Theorem 2.4,  $z_q(t; q)h$  coincides with the solution of the initial value problem

$$\begin{cases} \dot{v}_{q,h}(t) = Av_{q,h}(t) + G(q, t, v_{q,h}, h) + F_q(q, t, z(t; q))h \\ v_{q,h}(0) = 0. \end{cases}$$

where  $G(q, t, v, h) \doteq F_z(q, t, z(t; q))v + F_q(q, t, z(t; q))h$ . Now, let  $\mathcal{Q}_C$  be a compact subset of  $\mathcal{Q}$  and  $q_1, q_2 \in \mathcal{Q}_C$ . Then, for  $v \in Z_\delta$  and  $t \in [0, T]$ , there follows that

$$\begin{aligned} \|G(q_1, t, v, h) - G(q_2, t, v, h)\|_Z &\leq \|F_z(q_1, t, z(t; q_1)) - F_z(q_2, t, z(t; q_2))\|_{\mathcal{L}(Z_\delta, Z)} \|v\|_{Z_\delta} \\ &\quad + \|F_q(q_1, t, z(t; q_1)) - F_q(q_2, t, z(t; q_2))\|_{\mathcal{L}(\tilde{\mathcal{Q}}, Z)} \|h\|_{\tilde{\mathcal{Q}}} \\ &\leq L_1 \|z(t; q_1) - z(t; q_2)\|_{Z_\delta} \|v\|_{Z_\delta} + L_2 \|q_1 - q_2\|_{\tilde{\mathcal{Q}}} \|h\|_{\tilde{\mathcal{Q}}} \\ &\leq \left( \tilde{L}_1 \|v\|_{Z_\delta} + L_2 \|h\|_{\tilde{\mathcal{Q}}} \right) \|q_1 - q_2\|_{\tilde{\mathcal{Q}}}, \end{aligned}$$

where, in the third inequality the fact that  $z(t; q)$  is locally Lipschitz continuous with respect to  $q$  was used (Theorem 2.3). Therefore, the mapping  $q \rightarrow G(q, \cdot, v, h)$  is locally Lipschitz continuous and the Lipschitz constant can be chosen independent of  $v$  and of  $h$  on compact subsets of  $Z_\delta$  and  $\tilde{\mathcal{Q}}$ , respectively. Hence  $G(q, t, v, h)$  satisfies the hypothesis of Theorem 2.3 and the mapping  $q \rightarrow z_q(\cdot; q)h$  is locally Lipschitz continuous. Moreover, since the Lipschitz constant of  $G$  is independent of  $h$  on  $\tilde{\mathcal{Q}}$ -bounded sets, it follows immediately from the proof of Theorem 2.3 that the Lipschitz constant of the mapping  $q \rightarrow z_q(\cdot, q)h$  can also be chosen independent of  $h$  on  $\tilde{\mathcal{Q}}$ -bounded sets. The theorem is then proved.  $\blacksquare$

## 4. An application

In this section we consider an example in which parameter differentiability is proved in the following system of nonlinear partial differential equations:

$$\rho u_{tt} - \beta \rho u_{xxt} + \gamma u_{xxxx} = f(x, t) + (2\alpha_2(\theta - \theta_1)u_x - 4\alpha_4 u_x^3 + 6\alpha_6 u_x^5)_x, \quad x \in (0, 1), 0 \leq t \leq T \quad (4.1a)$$

$$C_v \theta_t - k \theta_{xx} = g(x, t) + 2\alpha_2 \theta u_x u_{xt} + \beta \rho u_{xt}^2, \quad x \in (0, 1), 0 \leq t \leq T \quad (4.1b)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in (0, 1) \quad (4.1c)$$

$$u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad 0 \leq t \leq T \quad (4.1d)$$

$$\theta_x(0, t) = \theta_x(1, t) = 0, \quad 0 \leq t \leq T. \quad (4.1e)$$

These equations arise from the conservation laws of linear momentum and energy in a one-dimensional shape memory body. The functions  $u$  and  $\theta$  represent transverse displacement and absolute temperature, respectively. Subscripts “ $x$ ” and “ $t$ ” denote partial derivatives and  $\rho, C_v, k, \beta, \gamma, \alpha_2, \alpha_4, \alpha_6, \theta_1$  are positive constants depending on the material being considered. The functions  $f(x, t)$  and  $g(x, t)$  denote distributed forces and distributed heat sources, respectively. For a detailed explanation of the model and the meaning of the parameters involved we refer the reader to [11] and the references therein.

We are interested in determining the differentiability of the solution of these equations with respect to the parameters  $\alpha_2, \alpha_4, \alpha_6$  and  $\theta_1$ .

The IBVP (4.1) can be written as an abstract nonlinear Cauchy Problem like (1.1) in an appropriate Banach Space. For this purpose let the admissible parameter set be defined as  $\mathcal{Q} \doteq \{q = (\alpha_2, \alpha_4, \alpha_6, \theta_1) \mid q \in \mathbb{R}_+^4\}$ , the state space  $Z$  as the Hilbert space  $H_0^1(0, 1) \cap H^2(0, 1) \times L^2(0, 1) \times L^2(0, 1)$  with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix} \right\rangle \doteq \gamma \int_0^1 u''(x) \overline{\tilde{u}''(x)} dx + \rho \int_0^1 v(x) \overline{\tilde{v}(x)} dx + \frac{C_v}{k} \int_0^1 \theta(x) \overline{\tilde{\theta}(x)} dx.$$

The operator  $A$  on  $Z$  is defined by

$$D(A) = \left\{ \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in Z \mid \begin{array}{l} u \in H^4(0, 1), \quad u(0) = u(1) = u''(0) = u''(1) = 0 \\ v \in H_0^1(0, 1) \cap H^2(0, 1) \\ \theta \in H^2(0, 1), \theta'(0) = \theta'(1) = 0 \end{array} \right\}$$

and for  $z = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in D(A)$ ,

$$A \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \doteq \begin{pmatrix} 0 & I & 0 \\ -\frac{\gamma}{\rho} D^4 & \beta D^2 & 0 \\ 0 & 0 & \frac{k}{C_v} D^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix},$$

where  $D^n \doteq \frac{\partial^n}{\partial x^n}$ .

Define also  $z_0(x) = \begin{pmatrix} u_0(x) \\ v_0(x) \\ \theta_0(x) \end{pmatrix}$  and  $F(q, t, z) : \mathcal{Q} \times [0, T] \times D \rightarrow Z$  by

$$F(q, t, z) = \begin{pmatrix} 0 \\ f_2(q, t, z) \\ f_3(q, t, z) \end{pmatrix},$$

where

$$\begin{aligned} \rho f_2(q, t, z)(x) &= f(x, t) + (2\alpha_2(\theta - \theta_1)u_x - 4\alpha_4u_x^3 + 6\alpha_6u_x^5)_x, \\ C_v f_3(q, t, z)(x) &= g(x, t) + 2\alpha_2\theta u_x v_x + \beta\rho v_x^2 \end{aligned}$$

and  $D = H_0^1(0, 1) \cap H^2(0, 1) \times H^1(0, 1) \times H^1(0, 1)$ .

With the above notation, the IBVP (4.1) takes the form

$$\begin{cases} \dot{z}(t) = Az(t) + F(q, t, z), & z(t) \in Z, \quad 0 \leq t \leq T, \\ z(0) = z_0. \end{cases} \quad (4.2)$$

Assume the following standing hypothesis.

**(H2).** For each fixed  $t \geq 0$ , the functions  $f(x, t)$ ,  $g(x, t)$  are in  $L^2(0, 1)$  and there exist nonnegative functions  $K_f(x)$ ,  $K_g(x) \in L^2(0, 1)$  such that

$$|f(x, t_1) - f(x, t_2)| \leq K_f(x)|t_1 - t_2|, \quad |g(x, t_1) - g(x, t_2)| \leq K_g(x)|t_1 - t_2|$$

for all  $x \in (0, 1)$ ,  $t_1, t_2 \in [0, T]$ .

The following result can be easily derived from theorems 3.7 and 3.11 in [12] with only slight modifications in order to take into account for the different boundary conditions being considered here. Since the modifications needed are trivial and not relevant for the goals pursued by this article, we do not give details here.

**Theorem 4.1.** ([12]) *Let the operator  $A$  and the mapping  $F$  be as defined above. Then  $A$  generates an analytic semigroup  $T(t)$  in  $Z$  and, if **(H2)** holds, then  $F$  satisfies **(H1)** for any  $\delta \in (\frac{3}{4}, 1)$  and by Theorem 2.1 has a unique classical solution  $z(t; q)$ .*

The following theorem and its corollary show that the operator  $A$  and the function  $F$  satisfy certain regularity conditions, which, in view of Theorems 2.4 and 3.1, ensure differentiability of the mapping  $q \rightarrow z(\cdot; q)$  and the Lipschitz continuity of its Fréchet derivative.

**Theorem 4.2.** *Let  $Z$ ,  $A$  and  $F(q, t, z)$  be as defined above and assume **(H2)** holds. Then the mapping  $(q, z(\cdot)) \rightarrow F(q, \cdot, z(\cdot))$  from  $\mathcal{Q} \times L^\infty(0, T : Z_\delta)$  into  $L^\infty(0, T : Z)$  is Fréchet differentiable in both variables. Also, the mappings  $(q, z(\cdot)) \rightarrow F_q(q, \cdot, z(\cdot))$  and  $(q, z(\cdot)) \rightarrow F_z(q, \cdot, z(\cdot))$  are locally Lipschitz continuous from  $\mathcal{Q} \times L^\infty(0, T : Z_\delta)$  into  $L^\infty(0, T : \mathcal{L}(\tilde{\mathcal{Q}}, Z))$  and from  $\mathcal{Q} \times L^\infty(0, T : Z_\delta)$  into  $L^\infty(0, T : \mathcal{L}(Z_\delta, Z))$ , respectively.*

*Proof.* It follows immediately that  $f_2(q, t, z)$  and  $f_3(q, t, z)$ , as previously defined, are differentiable with respect to both  $q$  and  $z$ , and their Fréchet derivatives are given by:

$$\begin{aligned} D_z f_2(q, t, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}) \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix} &= f_{2,u} \tilde{u} + f_{2,v} \tilde{v} + f_{2,\theta} \tilde{\theta}, \\ D_z f_3(q, t, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}) \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix} &= f_{3,u} \tilde{u} + f_{3,v} \tilde{v} + f_{3,\theta} \tilde{\theta}, \\ D_q f_2(q, t, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}) &= \frac{1}{\rho} \left[ 2\theta' u' + 2(\theta - \theta_1) u'', -12 (u')^2 u'', 30 (u')^4 u'', -2\alpha_2 u'' \right], \\ D_q f_3(q, t, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}) &= \frac{1}{C_v} [2\theta u' v', 0, 0, 0], \end{aligned}$$

where the linear operators  $f_{i,u}$ ,  $f_{i,v}$  and  $f_{i,\theta}$ ,  $i = 2, 3$  are

$$\begin{aligned} f_{2,u} &= \frac{1}{\rho} \left\{ 2\alpha_2 \theta' D + 2\alpha_2 (\theta - \theta_1) D^2 - 24\alpha_4 u' u'' D - 12\alpha_4 (u')^2 D^2 \right. \\ &\quad \left. + 120\alpha_6 (u')^3 u'' D + 30\alpha_6 (u')^4 D^2 \right\}, \\ f_{2,v} &= 0 \\ f_{2,\theta} &= \frac{1}{\rho} \{ 2\alpha_2 u' D + 2\alpha_2 u'' \}, \\ f_{3,u} &= \frac{1}{C_v} \{ 2\alpha_2 \theta v' D \} \\ f_{3,v} &= \frac{1}{C_v} \{ 2\alpha_2 \theta u' D + 2\beta \rho v' D \}, \\ f_{3,\theta} &= \frac{1}{C_v} \{ 2\alpha_2 u' v' \}. \end{aligned}$$

■

**Corollary 4.3.** *Under the same hypotheses of Theorem 4.2, the mapping  $q \rightarrow z(\cdot; q)$  is Fréchet differentiable and the mapping  $q \rightarrow z_q(\cdot; q)$  is locally Lipschitz continuous from  $\mathcal{Q}$  into  $L^\infty(0, T : \mathcal{L}(\tilde{\mathcal{Q}}, Z))$ .*

*Proof.* The proof is an immediate consequence of Theorems 4.2, 2.4 and 3.1. ■

## 5. Conclusions

In this article we have considered an abstract nonlinear evolution equation with an unknown parameter appearing in the nonlinear term. By employing the theory of analytic semigroups and a generalization of Gronwall's lemma for singular kernels we have derived sufficient conditions under which the solutions are differentiable with respect to the unknown parameter  $q$  with Lipschitz continuous Fréchet derivative. This condition is required for the convergence of the quasilinearization algorithms for identification of  $q$  from experimental data.



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