CONVERGENCE OF FINITE ELEMENTS ADAPTED FOR WEAK NORMS

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Abstract. We consider finite elements that are adapted to a (semi)norm that is weaker than the one of the trial space. We establish convergence of the finite element solutions to the exact one under the following conditions: refinement relies on unique quasi-regular element subdivisions and generates locally quasi-uniform grids; the finite element spaces are conforming, nested, and satisfy the inf-sup condition; the error estimator is reliable and appropriately locally efficient; the indicator of a non-marked element is bounded by the estimator contribution associated with the marked elements, and each marked element is subdivided at least once. This abstract convergence result is illustrated by two examples.

1. Introduction and Outline

Adaptivity has become a popular technique to increase the efficiency of finite element methods for boundary values problems. In practice, finite element grids are adapted to various error notions: the energy norm, other norms, or the output of certain functionals applied to the solution. However, the theoretical underpinning of the methods in terms of convergence and complexity results essentially restrict, up to now, to the most immediate cases of the energy norm and the norm of the trial space; see, e.g., [2, 6, 9, 10, 5, 8, 14].

This paper presents a basic convergence result for finite elements that are adapted to a (semi)norm that is possibly weaker than the one of the trial space. To this end, §2 gives general assumptions on the problem itself, the refinement framework, the finite element spaces, the approximate solution, the a posteriori error estimator, the marking strategy, and the step REFINE. They ensure the convergence of both error in the weaker (semi)norm and associated estimator. The proof is obtained by generalizing the convergence proof of [11] in a straight-forward manner.

In §3 we illustrate this convergence result by two examples: Lagrange elements for the Poisson problem that are adapted for the mean square error and Raviart-Thomas or Brezzi-Douglas-Marini elements in a mixed discretization that are adapted for the mean square error of the flux.

2. Abstract Convergence for Weak Norms

We first describe the problem class and adaptive algorithm and then present the convergence result.

2.1. **Problem Class and Error Notion.** We consider linear boundary value problems that can be reformulated in the following weak form: given a real Hilbert space $\mathbb{V}$ with norm $\|\cdot\|$, a continuous bilinear form $\mathcal{B} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$, and an element $f \in \mathbb{V}^*$ of the dual space of $\mathbb{V}$, find

$$(1) \qquad u \in \mathbb{V} : \qquad \mathcal{B}(u, w) = \langle f, w\rangle \qquad \forall w \in \mathbb{V}.$$

Key words and phrases. Adaptivity, conforming finite elements, convergence.

We suppose that the so-called *inf-sup (or Babuška-Brezzi) condition* holds: there exists $\alpha > 0$ such that

$$\inf_{\substack{v\in\mathbb{V}\\ \|v\|=1}} \sup_{\substack{w\in\mathbb{V}\\ \|w\|=1}} \mathcal{B}(v,w) \geq \alpha, \qquad \inf_{\substack{w\in\mathbb{V}\\ \|w\|=1}} \sup_{\substack{v\in\mathbb{V}\\ \|v\|=1}} \mathcal{B}(v,w) \geq \alpha. \tag{2a}$$

Concerning the error notion, we are interested in a seminorm $|\cdot|$ that is weaker than $\|\cdot\|$: there exists $C \geq 0$ such that

$$\forall\, v \in \mathbb{V} \quad |v| \leq C\|v\|. \tag{2b}$$

Below, we will introduce *'local features'* into (1) by making assumptions on a mesh-dependent counterpart of $|\cdot|$ and its interplay with $\mathcal{B}$. To this end, we suppose that $\mathbb{V}$ is a subspace of $L_p(\Omega;\mathbb{R}^m)$, where $p \in (1,\infty)$, $m \in \mathbb{N}$, and Ω is the underlying domain in $\mathbb{R}^d$, $d \geq 2$, that can be meshed. In what follows, we suppress the dependence on the data Ω, f, and $\mathcal{B}$.

2.2. **Adaptive Algorithm.** The adaptive algorithm for approximating u in (1) is an iteration of the following main steps:

(3)

1. $u_k := \mathsf{SOLVE}\big(\mathbb{V}(\mathcal{G}_k)\big)$.
2. $\{\mathcal{E}_k(E)\}_{E\in\mathcal{G}_k} := \mathsf{ESTIMATE}\big(u_k, \mathcal{G}_k\big)$.
3. $\mathcal{M}_k := \mathsf{MARK}\big(\{\mathcal{E}_k(E)\}_{E\in\mathcal{G}_k}, \mathcal{G}_k\big)$.
4. $\mathcal{G}_{k+1} := \mathsf{REFINE}\big(\mathcal{G}_k, \mathcal{M}_k\big)$, increment k.

In practice, a stopping test is used after step (2) for terminating the iteration; here we shall ignore it for notational convenience. The realization of these steps requires the following objects and modules:

Initial Grid and Framework for Refinement. An initial grid $\mathcal{G}_0$ of the domain Ω and a refinement procedure REFINE. The refinement procedure has two input arguments: a grid $\mathcal{G}$ and a subset $\mathcal{M} \subset \mathcal{G}$. All elements $E \in \mathcal{M}$ must be 'refined'. The input grid $\mathcal{G}$ can be the initial grid $\mathcal{G}_0$ or the output of a previous application of REFINE. A grid $\mathcal{G}'$ is called refinement of $\mathcal{G}$ whenever $\mathcal{G}'$ can be produced from $\mathcal{G}$ by a finite number of applications of REFINE. Initial grid and refinement procedure thus generate the set

$$\mathbb{G} := \{\mathcal{G} \mid \mathcal{G} \text{ is a refinement of } \mathcal{G}_0\}.$$

We shall write '$\preccurlyeq$' for '$\leq C$' where C may depend on data of (1), the class $\mathbb{G}$, and the modules ESTIMATE, MARK below, but not on a particular grid or the iteration number. Similarly, we say that some object is 'fixed' if it has the same dependencies.

We suppose that REFINE relies on *unique quasi-regular element subdivisions.* More precisely, there exist constants $q_1, q_2 \in (0,1)$ such that, irrespective of the grid $\mathcal{G}$, any element $E \in \mathcal{G}$ can be subdivided into $n(E) \geq 2$ subelements $E_1', \dots, E_{n(E)}'$ such that

$$E = E_1' \cup \dots \cup E_{n(E)}', \qquad |E| = |E_1'| + \dots + |E_{n(E)}'|, \tag{4a}$$

$$q_1|E| \leq |E_i'| \leq q_2|E|, \qquad i = 1,\dots,n(E), \tag{4b}$$

where $|E|$ stands for the d-dimensional Lebesgue measure of E.

These unique element subdivisions generate a 'master forest' $\mathcal{F}$ of infinite trees, where each node corresponds to an element, its direct successors to its subelements, and the roots to the elements of the initial grid $\mathcal{G}_0$. A subforest $\hat{\mathcal{F}} \subset \mathcal{F}$ is called finite if it has a finite number of nodes. Any finite tree may have interior nodes, i.e. nodes with successors, and does have leaf nodes, i.e. nodes without any successor.

Any subdivision $\mathcal{S}$ of the domain Ω that is subordinated to $\mathcal{G}_0$ is uniquely associated with a finite subforest $\mathcal{F}(\mathcal{S})$ of $\mathcal{F}$, where the leaf nodes are the elements of the subdivision. Given $n \in \mathbb{N}$ and a subset $\hat{\mathcal{S}}$ of such subdivision $\mathcal{S}$, we denote by $\mathcal{F}_n(\mathcal{S}, \hat{\mathcal{S}})$ the subforest of $\mathcal{F}$ that consists of $\mathcal{F}(\mathcal{S})$ and all successors of elements in $\hat{\mathcal{S}}$ up to generation n.

We suppose that the class $\mathbb{G}$ is a subclass of the subdivisions of Ω subordinated to $\mathcal{G}_0$ and is *locally quasi-uniform* in that

$$\sup_{\mathcal{G}\in\mathbb{G}} \max_{E\in\mathcal{G}} \#N_{\mathcal{G}}(E) \preccurlyeq 1, \qquad \sup_{\mathcal{G}\in\mathbb{G}} \max_{E'\in N_{\mathcal{G}}(E)} \frac{|E|}{|E'|} \preccurlyeq 1, \tag{4c}$$

where $N_{\mathcal{G}}(E) := \{E' \in \mathcal{G} \mid E' \cap E \neq \emptyset\}$ denotes the set of neighbors of E in $\mathcal{G}$. The grids in $\mathbb{G}$ may have additional properties like conformity.

Finite Element Spaces and Mesh-Dependent Norms. We suppose that the finite element spaces $\mathbb{V}(\mathcal{G})$, $\mathcal{G} \in \mathbb{G}$, are conforming, nested, and satisfy a discrete inf-sup condition: for any $\mathcal{G}, \mathcal{G}' \in \mathbb{G}$, there hold

$$\mathbb{V}(\mathcal{G}) \subset \mathbb{V} \tag{5a}$$

$$\mathcal{G}' \text{ is a refinement of } \mathcal{G} \implies \mathbb{V}(\mathcal{G}) \subset \mathbb{V}(\mathcal{G}') \tag{5b}$$

$$\inf_{\substack{v\in\mathbb{V}(\mathcal{G})\\ \|v\|=1}} \sup_{\substack{w\in\mathbb{V}(\mathcal{G})\\ \|w\|=1}} \mathcal{B}(v,w) \geq \beta \tag{5c}$$

with some fixed $\beta > 0$.

Moreover, we suppose that, for each grid $\mathcal{G} \in \mathbb{G}$, there is a pair $|\cdot|_{\mathcal{G}}$, $|\!|\!|\cdot|\!|\!|_{\mathcal{G}}$ of possibly *mesh-dependent seminorms* that is associated with the weak seminorm $|\cdot|$ of the error notion and has the following properties:

- $|\cdot|_{\mathcal{G}} = |\cdot|_{\mathcal{G};\Omega}$ is a seminorm on $\mathbb{V}$ that is close to $|\cdot|$, p-subadditive with respect to the domain, and absolutely continuous with respect to the Lebesgue measure in the following sense: for all $v \in \mathbb{V}$,

$$|v| \preccurlyeq |v|_{\mathcal{G}} \preccurlyeq \|v\|, \tag{5d}$$

$$\sum_{i=1}^{n} |v|^p_{\mathcal{G};\omega_i} \preccurlyeq |v|^p_{\mathcal{G};\Omega}, \tag{5e}$$

$$|v|_{\mathcal{G}_k;\Omega_k} \to 0, \tag{5f}$$

  where $\{\omega_i\}_{i=1}^n$ are disjoint subdomains of Ω, each one being a union of elements of $\mathcal{G}$, and $\{\Omega_k\}_k$ is a sequence of subdomains such that $|\Omega_k| \to 0$ and each Ω_k is a union of elements of a grid $\mathcal{G}_k$.
- $|\!|\!|\cdot|\!|\!|_{\mathcal{G}}$ is a seminorm on a subspace $\tilde{\mathbb{V}}(\mathcal{G})$ of $\mathbb{V}(\mathcal{G})$;
- the bilinear form is continuous with respect to the pair $|\cdot|_{\mathcal{G}}$, $|\!|\!|\cdot|\!|\!|_{\mathcal{G}}$ in a local sense: there is a constant $C_{\mathcal{B}} \geq 0$ such that, if ω is a union of elements of $\mathcal{G}$, then we have for any $v \in \mathbb{V}$ and any $w \in \tilde{\mathbb{V}}(\mathcal{G})$

$$w = 0 \text{ in } \Omega \setminus \omega \quad \implies \quad \mathcal{B}(v,w) \leq C_{\mathcal{B}} |v|_{\mathcal{G};\omega} \, |\!|\!|w|\!|\!|_{\mathcal{G}}\,. \tag{5g}$$

The role of these mesh-dependent seminorms will become clear from the example in §3.1.

SOLVE. We suppose that the output $u_{\mathcal{G}} := \mathsf{SOLVE}\big(\mathbb{V}(\mathcal{G})\big)$ is the *Galerkin approximation* of u in $\mathbb{V}(\mathcal{G})$:

$$u_{\mathcal{G}} \in \mathbb{V}(\mathcal{G}): \qquad \mathcal{B}(u_{\mathcal{G}}, w) = \langle f,\, w\rangle \qquad \forall w \in \mathbb{V}(\mathcal{G}). \tag{6}$$

Thanks to (5a) and (5c), the solution of (6) exists and is unique.

ESTIMATE. We suppose that $\{\mathcal{E}_{\mathcal{G}}(E)\}_{E\in\mathcal{G}} := \mathsf{ESTIMATE}(u_{\mathcal{G}}, \mathcal{G})$ has the following two properties for any grid $\mathcal{G} \in \mathbb{G}$: First, there holds the following *global upper bound* for the error in $|\cdot|$ of the Galerkin approximation $u_{\mathcal{G}}$:

$$|u_{\mathcal{G}} - u| \preccurlyeq \mathcal{E}_{\mathcal{G}}, \tag{7a}$$

where, given a subset $\hat{\mathcal{G}} \subset \mathcal{G}$, we define $\mathcal{E}_{\mathcal{G}}(\hat{\mathcal{G}}) := \left(\sum_{E\in\hat{\mathcal{G}}} \mathcal{E}^p_{\mathcal{G}}(E)\right)^{1/p}$ and set $\mathcal{E}_{\mathcal{G}} := \mathcal{E}_{\mathcal{G}}(\mathcal{G})$ and $\mathcal{E}_{\mathcal{G}}(\emptyset) := 0$.

Secondly, a *fixed finite subdivision depth implies a local lower bound with respect to a mesh-dependent dual seminorm of the residual.* More precisely, there is a fixed $n \in \mathbb{N}$ such that, for any element $E \in \mathcal{G}$ and any finer grid $\mathcal{G}' \in \mathbb{G}$ with $\mathcal{F}(\mathcal{G}') \supset \mathcal{F}_n\big(\mathcal{G}, N_{\mathcal{G}}(E)\big)$, there holds

$$\mathcal{E}_{\mathcal{G}}(E) \preccurlyeq \sup\Big\{\langle \mathcal{R}_{\mathcal{G}},\, w\rangle \mid w \in \tilde{\mathbb{V}}\big(\mathcal{G}'; \omega_{\mathcal{G}}(E)\big),\ \|w\|_{\mathcal{G}'} \le 1\Big\} + \mathrm{osc}_{\mathcal{G}}(E), \tag{7b}$$

where the oscillation indicator satisfies

$$\mathrm{osc}_{\mathcal{G}}(E) \preccurlyeq m(|E|)\Big(|u_{\mathcal{G}}|_{\mathcal{G};\omega_{\mathcal{G}}(E)} + \|D\|_{\mathbb{D}(\omega_{\mathcal{G}}(E))}\Big). \tag{7c}$$

Hereafter

- $\mathcal{R}_{\mathcal{G}} \in \mathbb{V}^*$ is the residual defined by

$$\langle \mathcal{R}_{\mathcal{G}},\, w\rangle := \mathcal{B}(u_{\mathcal{G}}, w) - \langle f,\, w\rangle, \qquad \forall w \in \mathbb{V}; \tag{8}$$

- $\omega_{\mathcal{G}}(E) \subset \Omega$ is the patch (union) of elements in $N_{\mathcal{G}}(E)$;
- $\tilde{\mathbb{V}}\big(\mathcal{G}'; \omega_{\mathcal{G}}(E)\big)$ is the space of 'local test functions' given by

$$\tilde{\mathbb{V}}\big(\mathcal{G}'; \omega_{\mathcal{G}}(E)\big) := \big\{w \in \tilde{\mathbb{V}}(\mathcal{G}') \mid w = 0 \text{ in } \Omega \setminus \omega_{\mathcal{G}}(E)\big\};$$

- $m : [0,\infty) \to [0,\infty)$ is a fixed, continuous, and nondecreasing function with $m(0) = 0$;
- $\mathbb{D}$ is another space with a norm that is p-subadditive and absolutely continuous with respect to the Lebesgue measure in the sense of (5e), (5f), and $D \in \mathbb{D}$ is given by the data of (1).

The global upper bound (7a) ensures that the error indicators do not overview any source of error. Inequality (7b) is the main step in proving a local lower error bound by Verfürth's constructive argument [15]: indeed, if one inserts (1) into (8) and recalls (5g), then (7b) readily yields the local lower error bound

$$\mathcal{E}_{\mathcal{G}}(E) \preccurlyeq |u_{\mathcal{G}} - u|_{\mathcal{G};\omega_{\mathcal{G}}(E)} + \mathrm{osc}_{\mathcal{G}}(E). \tag{9}$$

Thus, (7b) ensures, up to (7c) and the difference between $|\cdot|$ and $|\cdot|_{\mathcal{G}}$, the sharpness of the upper bound (7a) in a local sense. The presence of the oscillation indicator (7c) is discussed in Remark 4.7 of [11].

MARK. We suppose that the output $\mathcal{M} := \mathsf{MARK}\big(\{\mathcal{E}_{\mathcal{G}}(E)\}_{E\in\mathcal{G}}, \mathcal{G}\big)$ of marked elements has the property

$$\forall\, E \in \mathcal{G} \setminus \mathcal{M} \qquad \mathcal{E}_{\mathcal{G}}(E) \le \mathcal{E}_{\mathcal{G}}(\mathcal{M}). \tag{10}$$

We suppose (10) only for convenience; in §5 of [11] we consider a weaker condition that is sufficient and essentially necessary for convergence.

REFINE. We suppose that the output grid $\mathcal{G}' := \mathsf{REFINE}(\mathcal{G}, \mathcal{M})$ satisfies the *minimal requirement*

$$\mathcal{F}(\mathcal{G}') \supset \mathcal{F}_1(\mathcal{G}, \mathcal{M}), \tag{11}$$

that is, each marked element of the input grid is subdivided at least once in the output grid. Additional elements in $\mathcal{G} \setminus \mathcal{M}$ may be refined in order to fulfill (4c) or to ensure that the output grid is in the class $\mathbb{G}$.

2.3. **Convergence.** We now state the main result of this paper. The difference to Theorem 2.1 in [11] is that here the grids are adapted to the error in the seminorm $|\cdot|$, which is weaker than the one of the trial space.

Theorem 1 (Abstract Convergence for Weak Norms). *Let u be the exact solution of* (1)*, suppose that there holds* (2)*, and that $\{u_k\}_k$ is the sequence of approximate solutions generated by iteration* (3).

If the refinement framework, the finite element spaces, the modules SOLVE, ESTIMATE, MARK, and REFINE satisfy, respectively, (4)*,* (5)*,* (6)*,* (7)*,* (10)*, and* (11)*, then both error and estimator decrease to* 0*, that is*

$$|u_k - u| \to 0 \quad \textit{and} \quad \mathcal{E}_k \to 0 \quad \textit{as} \quad k \to \infty.$$

Proof. In view of [11, Lemma 4.2], $\{u_k\}_k$ converges to some $u_\infty \in \mathbb{V}$ and it remains to show that $u_\infty = u$. To this end, proceed as in [11, §4.2] with the following modifications: use (5g) instead of a 'local' continuity of $\mathcal{B}$ in terms of $\|\cdot\|$, sum p-powers of local (semi)norms instead of squares, and then exploit (5d) or (5f). $\square$

3. Two applications

The following two applications focus on the error notion, which is really weaker that the norm of the trial space; further examples are in [11, §3]. In what follows, $h_{\mathcal{G}}$ stands for the meshsize function associated with $\mathcal{G} \in \mathbb{G}$.

3.1. **Mean Square Error in Poisson's Problem.** We apply iteration (3) to generate finite element solutions to the Poisson problem that adaptively approach the exact solution in the L_2-error.

Problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded, polyhedral, and convex domain, set

$$\mathbb{V} = H_0^1(\Omega), \ \|\cdot\| = \|\nabla \cdot\|_{L_2(\Omega)}, \quad |\cdot| = \|\cdot\|_{L_2(\Omega)},$$
$$\mathcal{B}(v,w) = \int_\Omega \nabla v \cdot \nabla w, \quad v, w \in \mathbb{V},$$

and suppose $f \in L_2(\Omega)$. It is well known that there hold $f \in \mathbb{V}^*$ and (2).

Refinement framework. Let $\mathcal{G}_0$ be a suitable conforming triangulation of Ω into d-simplices and let $\mathbb{G}$ be the class of all triangulations that can be generated from $\mathcal{G}_0$ by iterative or recursive bisection; e.g. see [12]. Then (4) is fulfilled with $n(E) = 2$, and $q_1 = q_2 = \frac{1}{2}$; the hidden constants in (4c) depend on $\mathcal{G}_0$. Moreover, $\mathbb{G}$ is a shape-regular family of triangulations.

Finite element spaces and mesh-dependent norms. For any $\mathcal{G} \in \mathbb{G}$, we choose Lagrange elements of any fixed order ℓ,

$$\mathbb{V}(\mathcal{G}) := \mathbb{LE}_\ell(\mathcal{G}) \cap H_0^1(\Omega) := \{v \in H_0^1(\Omega) \mid \forall\, E \in \mathcal{G} \ v_{|E} \in \mathbb{P}_\ell(E)\},$$

which is contained in $\mathbb{V}$. Since coercivity and continuity are handed down to a restriction of $\mathcal{B}$ and spaces of piecewise polynomials are nested on nested grids, (5a)-(5c) are valid with $\beta = 1$.

Moreover, we define the mesh-dependent norms as follows: given any $v \in \mathbb{V}$ and any union ω of elements of $\mathcal{G}$, we set

$$|v|_{\mathcal{G};\omega} = \left(\sum_{E \subset \omega} \|v\|^2_{L_2(E)} + \|h_{\mathcal{G}}^{1/2} v\|^2_{L_2(\partial E)} \right)^{1/2} \tag{12a}$$

and, for any $w \in \tilde{\mathbb{V}}(\mathcal{G}) = \mathbb{V}(\mathcal{G})$,

$$|\!|\!| w |\!|\!|_{\mathcal{G}} = \left(\sum_{E \in \mathcal{G}} \|D^2 w\|^2_{L_2(E)} + \|h_{\mathcal{G}}^{-1/2} \partial_n w\|^2_{L_2(\partial E)} \right)^{1/2}. \tag{12b}$$

Then $|\cdot|_{\mathcal{G};\Omega}$ is a norm on $\mathbb{V}$ and, in view of the scaled trace theorem $\|\cdot\|_{L_2(\partial E)} \preccurlyeq \|h_{\mathcal{G}}^{-1/2}\cdot\|_{L_2(E)} + \|h_{\mathcal{G}}^{1/2}\nabla\cdot\|_{L_2(E)}$ and the Poincaré inequality, (5d), (5e), and (5f) are valid. Moreover, $|\!|\!|\cdot|\!|\!|_{\mathcal{G}}$ is a norm on $\mathbb{V}(\mathcal{G})$ and (5g) is readily verified after an element-wise integration by parts.

Approximate solution and estimator. We suppose that SOLVE outputs the Galerkin approximation given by (6). Given such Galerkin solution $u_{\mathcal{G}}$ on a grid $\mathcal{G}$, the output of $\mathsf{ESTIMATE}$ is the standard residual estimator $\{\mathcal{E}_{\mathcal{G}}(E)\}_{E\in\mathcal{G}}$ for the $L_2(\Omega)$-error given by

$$\mathcal{E}_{\mathcal{G}}^2(E) := \|h_{\mathcal{G}}^{3/2}\,[\![\partial_n u_{\mathcal{G}}]\!]\,\|_{L_2(\partial E\setminus\partial\Omega)}^2 + \|h_{\mathcal{G}}^2(f+\Delta u_{\mathcal{G}})\|_{L_2(E)}^2, \quad E\in\mathcal{G},$$

where $[\![\partial_n u_{\mathcal{G}}]\!]$ stands for the jump of the normal derivative of $u_{\mathcal{G}}$ across interelement sides. This estimator fulfills (7) with

$$n = \begin{cases} 3 & \text{if } d=2,\\ 6 & \text{if } d=3,\end{cases} \qquad \operatorname{osc}_{\mathcal{G}}(E) = \|h_{\mathcal{G}}(f-\bar f_{\mathcal{G}})\|_{L_2(\omega_{\mathcal{G}}(E))},$$
$$m(s) = s^{1/d},\ s\in[0,\infty), \quad \mathbb{D} = L_2(\Omega), \quad D = f,$$

where $\bar f_{\mathcal{G}}$ is the L_2-projection of f on the space of possibly discontinuous piecewise polynomials of degree $\le \ell-1$; indeed, for (7a) see [15, Prop. 3.8] and for (7b) see [10, §6] but use (5g) with the mesh-dependent norms (12).

Marking strategy and refinement rule. Take any marking strategy ensuring that the biggest indicator is marked and require only that each marked simplex is bisected at least once. Then (10) and (11) are valid.

Under the above assumptions, Theorem 1 ensures that

$$\|u_k - u\|_{L_2(\Omega)} \to 0 \quad \text{and} \quad \mathcal{E}_k \to 0 \quad \text{as} \quad k\to\infty.$$

To our best knowledge, this is the first convergence result for the Poisson problem where the adaptation is not directed by an energy norm estimator.

3.2. **Mean Square Error of the Flux in Mixed Discretizations.** For mixed discretizations of the Poisson problem, we consider iteration (3) with an estimator for the approximation error in the flux.

Problem. Let Ω be a bounded, connected, polyhedral Lipschitz domain in $\mathbb{R}^2$. The mixed formulation of Poisson's problem and the error notion are given by

$$\mathbb{V} = \mathbf{V}\times\mathbb{Q} \quad \text{with} \quad \mathbf{V} = H(\operatorname{div};\Omega),\ \mathbb{Q} = L_2(\Omega)$$
$$\|v\|_{\mathbb{V}}^2 = \|\boldsymbol{v}\|_{L_2(\Omega;\mathbb{R}^2)}^2 + \|\operatorname{div}\boldsymbol{v}\|_{L_2(\Omega)}^2 + \|q\|_{L_2(\Omega)}^2, \quad |v| = \|\boldsymbol{v}\|_{L_2(\Omega;\mathbb{R}^2)}$$
$$\mathcal{B}(v,w) = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{w} - \int_\Omega q\operatorname{div}\boldsymbol{w} + \int_\Omega \operatorname{div}\boldsymbol{v}\, r,$$

for $v = [\boldsymbol{v},q]$, $w = [\boldsymbol{w},r]\in\mathbb{V}$. Suppose $f\in L_2(\Omega)$, which is identified with $(\mathbf{0},f)\in\mathbb{V}^*$. Then (2) is valid; see Example 1.2 in [3, §II.1.2].

Refinement framework, finite element spaces, and seminorms. We use the same refinement framework as in §3.1 for $d=2$ and choose Raviart-Thomas or Brezzi-Douglas-Marini elements of order ℓ or $\ell+1$ for the flux variable and piecewise polynomials of degree $\le\ell$ for the scalar variable: given a triangulation $\mathcal{G}\in\mathbb{G}$, we set

$$\mathbb{V}(\mathcal{G}) = \mathbf{V}(\mathcal{G})\times\mathbb{Q}(\mathcal{G}) \quad \text{with} \quad \mathbf{V}(\mathcal{G}) = \mathbb{RT}_\ell(\mathcal{G}) \text{ or } \mathbb{BDM}_\ell(\mathcal{G}),$$

where

$$\mathbb{Q}(\mathcal{G}) := \left\{ q \in L_2(\Omega) \mid \forall E \in \mathcal{G}\ q_{|E} \in \mathbb{P}_\ell(E) \right\},$$
$$\mathbb{RT}_\ell(\mathcal{G}) := \left\{ \boldsymbol{w} \in H(\operatorname{div};\Omega) \mid \forall E \in \mathcal{G}\ \boldsymbol{w}_{|E} \in \left(\mathbb{P}_\ell(E;\mathbb{R}^2) + \boldsymbol{x}\,\mathbb{P}_\ell(E)\right) \right\},$$
$$\mathbb{BDM}_\ell(\mathcal{G}) := \left\{ \boldsymbol{w} \in H(\operatorname{div};\Omega) \mid \forall E \in \mathcal{G}\ \boldsymbol{w}_{|E} \in \mathbb{P}_{\ell+1}(E;\mathbb{R}^2) \right\}.$$

In both cases, the inclusion $\operatorname{div} \mathbf{V}(\mathcal{G}) \subset \mathbb{Q}(\mathcal{G})$ and (5a)-(5c) hold; see Prop. 1.1 in [3, §IV.1.2].

Moreover, we let $|\cdot|_{\mathcal{G}} = |\cdot|$, which does not depend on $\mathcal{G}$. Then (5d)-(5f) are valid; the seminorm $\vert\!\vert\!\vert\cdot\vert\!\vert\!\vert_{\mathcal{G}}$ will be chosen below.

Approximate solution and estimator. Let SOLVE output the Galerkin solution of (6) and, writing $u_{\mathcal{G}} = [\boldsymbol{u}_{\mathcal{G}}, p_{\mathcal{G}}]$, we suppose that $\mathsf{ESTIMATE}$ outputs $\{\mathcal{E}_{\mathcal{G}}(E)\}_{E\in\mathcal{G}}$ given by

$$\mathcal{E}_{\mathcal{G}}^2(E) = \|h_{\mathcal{G}} \operatorname{rot} \boldsymbol{u}_{\mathcal{G}}\|_{L_2(E)}^2 + \|h_{\mathcal{G}}^{1/2} [\![\boldsymbol{u}_{\mathcal{G}} \cdot \boldsymbol{t}]\!] \|_{L_2(\partial E\cap\Omega)}^2 + \|h_{\mathcal{G}}(\bar f_{\mathcal{G}} - f)\|_{L_2(E)}^2,$$

where $\operatorname{rot} \boldsymbol{v} = \partial_{x_2} v_1 - \partial_{x_1} v_2$, $\bar f_{\mathcal{G}}$ stands for the $L_2(\Omega)$-orthogonal projection of f onto $\mathbb{Q}(\mathcal{G})$, and on any inter-element side, $\boldsymbol{t}$ stands for a fixed unit tangent vector. We shall prove that this estimator satisfies (7) with

$$n = 4, \quad m(s) = s^{1/d},\ s \in [0,\infty), \quad \operatorname{osc}_{\mathcal{G}}(E) = \|h_{\mathcal{G}}(\bar f_{\mathcal{G}} - f)\|_{L_2(\omega_{G}(E))},$$
$$\mathbb{D} = L_2(\Omega),\ \text{and } D = f.$$

Before embarking on the proper proof of the a posteriori bounds, we recall the orthogonal Helmholtz-decomposition [7, Theorem III.3.2]:

$$L_2(\Omega;\mathbb{R}^2) = \nabla H^1(\Omega)/\mathbb{R} \oplus \mathbf{curl}\, H^1_{\partial\Omega}(\Omega), \tag{13}$$

where $H^1_{\partial\Omega}(\Omega)$ denotes the space of all $H^1(\Omega)$-functions that are constant on each connected component of $\partial\Omega$ and $\mathbf{curl}\,\phi = \left[-\partial_{x_2}\phi, \partial_{x_1}\phi\right]^T$, which has rot as adjoint operator. Note that $\phi \in H^1_{\partial\Omega}(\Omega)$ implies $\mathbf{curl}\,\phi\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. The decomposition (13) appears in the relationship of error $\boldsymbol{u}_{\mathcal{G}} - \boldsymbol{u}$ and residual $\mathcal{R}$: If $w = [\mathbf{0}, -\psi] \in \mathbb{V}$ with $\psi \in \nabla H^1(\Omega)/\mathbb{R}$ normalized such that $\int_{\partial\Omega}(\boldsymbol{u}_{\mathcal{G}} - \boldsymbol{u})\cdot\boldsymbol{n}\psi = 0$, then

$$\int_\Omega (\boldsymbol{u}_{\mathcal{G}} - \boldsymbol{u})\cdot\nabla\psi = \mathcal{B}(u_{\mathcal{G}} - u, w) = \langle \mathcal{R}_{\mathcal{G}},\, w\rangle = \int_\Omega (\bar f_{\mathcal{G}} - f)\psi \tag{14}$$

thanks to integration by parts, (1), and (6). Moreover, if $w = [\mathbf{curl}\,\phi, 0] \in \mathbb{V}$ with $\phi \in H^1_{\partial\Omega}(\Omega)$, then

$$\int_\Omega (\boldsymbol{u}_{\mathcal{G}} - \boldsymbol{u})\cdot(\mathbf{curl}\,\phi) = \mathcal{B}(u_{\mathcal{G}} - u, w) = \langle \mathcal{R}_{\mathcal{G}},\, w\rangle = \int_\Omega \boldsymbol{u}_{\mathcal{G}}\cdot\mathbf{curl}\,\phi \tag{15}$$

because $\mathbf{curl}\,\phi$ is divergence-free and again thanks to (1).

The proof of the upper bound (7a) can be established by exploiting both the relationship (14) for the gradient part and (15) for the **curl**-part of the error; proceed similarly to the proof of [1, Theorem 3.1] and notice that, for $\phi \in H^1_{\partial\Omega}(\Omega)$, the interpolation operator in [13] allows to choose an approximation $\phi_{\mathcal{G}}$ that equals ϕ on $\partial\Omega$ and, thus, the estimator does not contain contributions on $\partial\Omega$.

We now derive the discrete local lower bound (7b), which appears to be new. Since $\|h_{\mathcal{G}}(\bar f_{\mathcal{G}} - f)\|_{L_2(E)}$ appears in the oscillation indicator, we only have to deal with terms that are related to (15). This suggests to construct discrete functions $\mathbf{curl}\,\phi \in \mathbf{V}(\mathcal{G}')$ for a suitable refinement $\mathcal{G}'$ of $\mathcal{G}$. To this end, we employ the Lagrange elements $\mathbb{LE}_{\ell+1}(\mathcal{G}')$ of §3.1. Given a subdomain $\omega \subset \Omega$, set $\mathbb{LE}_{\ell+1}(\mathcal{G}';\omega) := \mathbb{LE}_{\ell+1}(\mathcal{G}') \cap H^1_0(\omega)$. Since continuity of $\phi \in \mathbb{LE}_{\ell+1}(\mathcal{G}')$ across interelement edges entails continuity of $\mathbf{curl}\,\phi\cdot\boldsymbol{n}$ across those edges and $\mathbf{curl}\,\phi$ is element-wise a polynomial of degree $\leq \ell$, we have $\mathbf{curl}\,\mathbb{LE}_{\ell+1}(\mathcal{G}';\omega) \subset \mathbf{V}(\mathcal{G}';\omega)$ for any union ω of elements in $\mathcal{G}$. This motivates to use $\tilde{\mathbb{V}}(\mathcal{G}') = [\mathbf{curl}\,\mathbb{LE}_{\ell+1}(\mathcal{G}'), 0]$

and we choose $\|\!|\cdot|\!\|_{\mathcal{G}} = |\cdot|$, which does not depend on $\mathcal{G}$. Thanks to (15) we obtain (5g).

To bound $\|h_{\mathcal{G}} \operatorname{rot} \boldsymbol{u}_{\mathcal{G}}\|_{L_2(E)}$ for a given element $E \in \mathcal{G}$, we now use a variant of Verfürth's constructive argument. We subdivide E by 3 bisections, thus creating a node inside E. Let λ_E be the continuous piecewise affine hat function associated with that node. Testing (15) with $v = [\mathbf{curl}\, \phi, 0]$ where $\phi = \lambda_E \operatorname{rot} \boldsymbol{u}_{\mathcal{G}} \in \mathbb{LE}_{\ell+1}(\mathcal{G}'; E)$ and standard scaling arguments then yield the desired bound. To proceed similarly for the remaining jump indicators, we need the following technical lemma.

Lemma 2. *Let S be an interval, divided into four subintervals $\mathcal{G}_S = \{S_1, \dots, S_4\}$ of same size, and let P_S be the $L_2(S)$-orthogonal projection onto $\mathbb{LE}_{\ell+1}(\mathcal{G}_S) \cap H_0^1(S)$. Then $\|J\|^2_{L_2(S)} \preccurlyeq \int_S J P_S J$ for all $J \in \mathbb{P}_{\ell+1}(S)$.*

Proof. Thanks to a standard scaling argument, we only have to prove the claim for the interval $S = (0, 1)$, decomposed by the points $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

$\boxed{1}$ We first show that, for any $J \in \mathbb{P}_{\ell+1}(S)\setminus\{0\}$, there exists a $\phi \in \mathbb{B} := \mathbb{LE}_{\ell+1}(\mathcal{G}_S) \cap H_0^1(S)$ with $\int_S J\phi \neq 0$. Suppose this is not the case, i.e. there is a $J \in \mathbb{P}_{\ell+1}(S) \setminus \{0\}$ such that $\int_S J\phi = 0$ for all $\phi \in \mathbb{B}$.

Let ϕ_1 be the continuous piecewise affine hat function at $\frac{1}{4}$. In view of our assumption, we have

$$\forall q \in \mathbb{P}_\ell(S) \qquad \int_0^{\frac{1}{2}} J q \phi_1 = \int_S J q \phi_1 = 0.$$

Since $\phi_1 > 0$ on $(0, \frac{1}{2})$, the left hand side defines weighted scalar product on $L_2(0, \frac{1}{2})$. Hence J has $\ell + 1$ roots in $(0, \frac{1}{2})$. The same argument shows that J has also $\ell + 1$ roots in $(\frac{1}{2}, 1)$. Since $J \in \mathbb{P}_{\ell+1}(S)$ and has $2\ell + 2 > \ell + 1$ roots in $(0, 1)$, it has to vanish, which is a contradiction.

$\boxed{2}$ Thanks to step 1, the $L_2(S)$-orthogonal projection P_S verifies $P_S J \neq 0$ for all $J \in \mathbb{P}_{\ell+1} \setminus \{0\}$. Consequently, the continuity of P_S gives

$$\min_{\|J\|_{L_2(S)}=1} \int_S J P_S J = \min_{\|J\|_{L_2(S)}=1} \|P_S J\|^2_{L_2(S)} = \alpha > 0,$$

which directly implies $\|J\|^2_{L_2(S)} \leq \alpha^{-1} \int_S J P_S J$ for all $J \in \mathbb{P}_{\ell+1}(S)$. $\square$

To bound $\|h_{\mathcal{G}}^{1/2} [\![\boldsymbol{u}_{\mathcal{G}} \cdot \boldsymbol{t}]\!]\|_S$ for a given interelement side $S = E \cap E'$, we bisect E and E' four times, entailing a subdivision of S into four subintervals of same size. Testing (15) with $v = [\mathbf{curl}\, \phi, 0]$ where ϕ is an extension of $P_S([\![\boldsymbol{u}_{\mathcal{G}} \cdot \boldsymbol{t}]\!]_S)$ to $\mathbb{LE}_{\ell+1}(\mathcal{G}'; E \cup E')$, Lemma 2, and standard arguments then conclude the proof of (7b).

Marking strategy and refinement rule. We make the same assumptions on marking strategy and refinement rule in §3.1.

Under the above assumptions, Theorem 1 ensures that

$$\|\boldsymbol{u}_k - \boldsymbol{u}\|_{L_2(\Omega)} \to 0 \quad \text{and} \quad \mathcal{E}_k \to 0 \quad \text{as} \quad k \to \infty.$$

This generalizes the convergence result of Carstensen and Hoppe [4] to Raviart-Thomas and Brezzi-Douglas-Marini elements of any fixed order.

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